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On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Divisions of a Scale. *By* W. F. SHEPPARD, M.A., LL.M. Read March 10th, 1898. Received, in revised form, May 10th, 1898.

I. GENERAL FORMULÆ.

1. Let $$z = f(x)$$

be the equation to a curve which lies wholly on the positive side of the axis of x, and extends continuously from the ordinate $z_0 = f(x_0)$ to the ordinate $z_p = f(x_p)$, and which is subject to the further condition that $f(x)$ is single-valued between $x = x_0$ and $x = x_p$. Let the range $x_p - x_0$ be divided into p equal segments, each equal to h, by points whose abscissæ are $x_1, x_2, \ldots, x_{p-1}$, so that

$$x_r = x_0 + rh;$$

let the ordinates to the curve from the points of division be $z_1, z_2, \ldots, z_{p-1}$, so that

$$z_r = f(x_r);$$

and let A_r denote the area between two consecutive ordinates z_{r-1} and z_r, the curve, and the base $z = 0$, so that

$$A_r = \int_{x_{r-1}}^{x_r} f(x)\,dx.$$

For the mathematical treatment of statistics we require a convenient formula for calculating, approximately, the value of $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$, where—

(i.) $\phi(x)$ is a known function of x, which is single-valued, finite, and continuous from $x = x_0$ to $x = x_p$; and

(ii.) the form of $f(x)$ is unknown, but either

(a) the values of the isolated ordinates $z_0 = f(x_0)$, $z_1 = f(x_1)$, ..., $z_p = f(x_p)$ are known, or

(b) the values of the successive areas

$$A_1 = \int_{x_0}^{x_1} f(x)\,dx, \quad A_2 = \int_{x_1}^{x_2} f(x)\,dx, \quad \ldots, \quad A_p = \int_{x_{p-1}}^{x_p} f(x)\,dx$$

are known.

2. The Euler-Maclaurin formula

$$\int u_x\,dx = C+\Sigma u_x-\tfrac{1}{2}u_x-\frac{B_1}{2!}\frac{du_x}{dx}+\frac{B_2}{4!}\frac{d^3u_x}{dx^3}-\dots \qquad (1)^*$$

may be made to meet both these cases. For the first case we have at once

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$=\tfrac{1}{2}f(x_0)\,\phi(x_0)+f(x_1)\,\phi(x_1)+\dots+f(x_{p-1})\,\phi(x_{p-1})+\tfrac{1}{2}f(x_p)\,\phi(x_p)$$

$$-\left[\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\dots\right\}f(x)\,\phi(x)\right]_{x=x_0}^{x=x_p} \qquad (2)$$

$$=\tfrac{1}{2}z_0\varphi(x_0)+z_1\phi(x_1)+\dots+z_{p-1}\phi(x_{p-1})+\tfrac{1}{2}z_p\varphi(x_p)$$

$$+\tfrac{1}{12}(\Delta-\tfrac{1}{2}\Delta^2+\tfrac{1}{60}\Delta^3-\dots)f(x_0)\,\phi(x_0)$$

$$+\tfrac{1}{12}(\Delta'-\tfrac{1}{2}\Delta'^2+\tfrac{1}{60}\Delta'^3-\dots)f(x_p)\,\phi(x_p), \qquad (2A)$$

the second set of differences $\Delta', \Delta'^2, \Delta'^3, \dots$ being accented to show that they are taken backwards, *i.e.*, from $x = x_p$ towards $x = x_0$.

For the second case we write

$$F(x)=\int_x^{x_p} f(x)\,dx,$$

so that $F(x)$ denotes the whole area of the curve lying beyond the ordinate $z=f(x)$. Then

$$\left.\begin{aligned}
F(x_0) &= A_1+A_2+A_3+\dots+A_p\\
F(x_1) &= \quad A_2+A_3+\dots+A_p\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
F(x_{p-1}) &= \qquad\qquad A_p\\
F(x_p) &= \qquad\qquad 0
\end{aligned}\right\}, \qquad (3)$$

so that $F(x)$ is a function for which the isolated values $F(x_0)$, $F(x_1), \dots, F(x_p)$ are known. By integration by parts,

$$\int_{x_0}^{x_p} f(x)\,\varphi(x)\,dx = F(x_0)\,\phi(x_0)+\int_{x_0}^{x_p} F(x)\,\phi'(x)\,dx, \qquad (4)\dagger$$

---

* Here Σu_x denotes the sum up to and including u_x. If Σ is used as equivalent to Δ^{-1}, Σu_x is the sum up to and including u_{x-1}, and we must replace $-\frac{1}{2}u_x$ by $+\frac{1}{2}u_x$.

† For instance, let $F(x)$ denote the proportion of individuals alive at age x for every one alive at age x_0; then $f(x)dx$ will denote the proportion who die between ges $x-\frac{1}{2}dx$ and $x+\frac{1}{2}dx$. If

$$\phi(x) = P(1+i)^{-(x-x_0)},$$

and therefore

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \frac{1}{h}F(x_0)\,\phi(x_0) + \tfrac{1}{2}F(x_0)\,\phi'(x_0) + F(x_1)\,\phi'(x_1) + \ldots + F(x_{p-1})\,\phi'(x_{p-1})$$

$$- \left[\left\{\frac{B_1}{2!}h\frac{d}{dx} - \frac{B_2}{4!}h^3\frac{d^3}{dx^3} + \ldots\right\} F(x)\,\phi'(x)\right]_{x=x_0}^{x=x_p} \qquad (5)$$

$$= \frac{1}{h}F(x_0)\,\phi(x_0) + \tfrac{1}{2}F(x_0)\,\phi'(x_0) + F(x_1)\,\phi'(x_1) + \ldots + F(x_{p-1})\,\phi'(x_{p-1})$$

$$+ \tfrac{1}{12}(\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{60}\Delta^3 - \ldots)\,F'(x_0)\,\phi'(x_0)$$

$$+ \tfrac{1}{12}(\Delta - \tfrac{1}{2}\Delta'^2 + \tfrac{1}{60}\Delta'^3 - \ldots)\,F(x_p)\,\phi'(x_p). \qquad (5A)$$

3. The formulæ (2A) and (5A) are sufficient, but they involve the calculation of successive differences. In the cases which we have specially in view the curve $z = f(x)$ touches the base $z = 0$, to a very high order of contact, at the extreme points $x = x_0$ and $x = x_p$. In such a case $f(x_0)$ and $f(x_p)$ and their first few differential coefficients are zero. Hence the expression given by the square brackets in (2) is negligible, and we have, *approximately*,

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx = \tfrac{1}{2}z_0\phi(x_0) + z_1\phi(x_1) + \ldots + z_{p-1}\phi(x_{p-1}) + \tfrac{1}{2}z_p\phi(x_p). \qquad (2B)$$

Also the first few differential coefficients of $F(x_0)$ and $F(x_p)$ are zero, and

$$F(x_p) = 0;$$

but $F(x_0)$ is not zero. Hence the expression given by the square brackets in (5) is negligible, except as to the part involving $F(x_0)$. Taking this out, we have, approximately,

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \tfrac{1}{2}F(x_0)\,\phi'(x_0) + F(x_1)\,\phi'(x_1) + \ldots + F(x_{p-1})\,\phi'(x_{p-1})$$

$$+ \frac{1}{h}F(x_0)\left\{\phi(x_0) + \frac{B_1}{2!}h^2\phi''(x_0) - \frac{B_2}{4!}h^4\phi^{iv}(x_0) + \ldots\right\} \qquad (5B)$$

$\int_{x_0}^{\infty} f(x)\,\phi(x)\,dx$ will be the present value of a reversion of value P, due on the death of a person aged x_0, interest being reckoned at 100i per cent. per annum, while $\int_{x_0}^{\infty} F(x)\,\phi(x)\,dx$ will be the present value of a continuous annuity at the rate of P per annum, payable to a person of that age, and (4) shows that

(present value of reversion) $= P - \log_e(1+i)$. (present value of annuity),

which is a well-known formula.

the expression omitted from the right-hand side being

$$\left[F(x)\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\ldots\right\}\varphi'(x)\right.$$
$$\left.-\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\ldots\right\}F(x)\varphi'(x)\right]_{x=x_0}^{x=x_p}. \qquad (5\text{C})$$

The formula (2B) is convenient for calculation, but (5B) is obviously inconvenient, on account of its unsymmetrical form. Our object is therefore to obtain a general formula for $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$, which shall be better suited for these particular cases.

4. Let $\xi_1, \xi_2, \ldots, \xi_p$ be the abscissæ of the middle points of the bases of the areas $A_1, A_2, \ldots, A_p$, so that

$$\xi_r = \tfrac{1}{2}(x_{r-1}+x_r).$$

We wish to obtain the value of $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$ in the form

$$\Sigma A_r \phi_1(\xi_r) + h\left[\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

where $\phi_1(x)$ is a function of x which can be derived from $\phi(x)$, and $\phi_2(x)$ is an expression which can be derived from $f(x)$ and $\phi(x)$, and which may be disregarded when $f(x)$ and its first few differential coefficients are zero. We must therefore express

$$\int_{x_{r-1}}^{x_r} f(x)\,\phi(x)\,dx \equiv \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)\,\phi(\xi_r+\theta)\,d\theta$$

in the form

$$\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)\,d\theta\,.\,\phi_1(\xi_r) + h\left[\phi_2(x)\right]_{x=x_{r-1}}^{x=x_r};$$

in other words, we must find such a value for $\phi_1(\xi)$ that

$$\Phi(\xi) \equiv \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,\phi(\xi+\theta)\,d\theta - \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,d\theta\,.\,\phi_1(\xi) \qquad (6)$$

may be capable of being expressed in the form

$$h\left\{\phi_2(\xi+\tfrac{1}{2}h)-\phi_2(\xi-\tfrac{1}{2}h)\right\}, \qquad (7)$$

where $\phi_2(\xi)$ contains only $f(\xi)$ and $\phi(\xi)$ and their successive differential coefficients.

Now, in order that $\Phi(\xi)$ may be expressible in the form (7), it is

necessary and sufficient (subject to certain conditions as to convergence) that $\Phi(\xi)\,d\xi$ should be an exact differential. It is necessary, for if we expand (7) in ascending powers of h we have only differential coefficients of $\phi_2(\xi)$. And it is sufficient, for if $\Phi(\xi)$ is equal to $h\Psi'(\xi)$ we can obviously choose the coefficients $C_1, C_2, \ldots$ so as to make it equal to

$$\left\{\Psi(\xi+\tfrac{1}{2}h)-\Psi(\xi-\tfrac{1}{2}h)\right\}+C_1\left\{\Psi''(\xi+\tfrac{1}{2}h)-\Psi''(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^3 + C_2\left\{\Psi^{iv}(\xi+\tfrac{1}{2}h)-\Psi^{iv}(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^4+\ldots.$$

The values of these coefficients can be obtained most simply by the method of operators, which shows that

$$h\Psi'(\xi) = (e^{\frac{1}{2}h(d/d\xi)}-e^{-\frac{1}{2}h(d/d\xi)})\,\frac{\frac{1}{2}h\frac{d}{d\xi}}{\sinh\frac{1}{2}h\frac{d}{d\xi}}\,\Psi(\xi)$$

$$= \left\{\Psi(\xi+\tfrac{1}{2}h)-\Psi(\xi-\tfrac{1}{2}h)\right\} - \frac{P_1}{2!}\left\{\Psi''(\xi+\tfrac{1}{2}h)-\Psi''(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^3 + \frac{P_2}{4!}\left\{\Psi^{iv}(\xi+\tfrac{1}{2}h)-\Psi^{iv}(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^4 - \ldots, \tag{8}$$

where $P_1, P_2, \ldots$ are the coefficients in the expansion

$$\frac{\theta}{\sinh\theta} = 1-\frac{P_1}{2!}\theta^2+\frac{P_2}{4!}\theta^4-\ldots, \tag{9}$$

so that

$$P_r = (2^{2r}-2)\,B_r. \tag{10}$$

We consider that we are only dealing with cases in which the series (8) is initially convergent.

Now let us assume that

$$\phi_1(\xi) = \psi\left(\tfrac{1}{2}h\frac{d}{d\xi}\right)\phi(\xi). \tag{11}$$

Then if we denote $\frac{1}{2}h\frac{d}{d\xi}$ operating on $f(\xi)$ by D_1, and $\frac{1}{2}h\frac{d}{d\xi}$ operating on $\phi(\xi)$ by D_2, $\Phi(\xi)$ is equal to

$$h\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\psi(D_2)\right\}f(\xi)\,\phi(\xi). \tag{12}$$

In order that $\Phi(\xi)\,d\xi$ may be an exact differential, it is clearly necessary that (12) may be capable of being written in the form

$$h(D_1+D_2)\,\chi(D_1,\,D_2)\,f(\xi)\,\phi(\xi), \tag{13}$$

where $\chi(D_1,\,D_2)$ is a function containing only positive integral powers (including the power 0) of D_1 and D_2. It will be seen that

$$\psi(D_2)=\frac{D_2}{\sinh D_2}$$

satisfies these conditions. For

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}$$

can be expanded so as to include only positive integral powers of D_1 and D_2; and it vanishes when $D_1+D_2=0$, so that D_1+D_2 is a factor. Hence we have

$$\int_{x_0}^{x_1} f(x)\,\phi(x)\,dx = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,\phi(\xi+\theta)\,d\theta$$

$$=\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,d\theta\,.\left\{\phi(\xi)-\frac{P_1}{2!}\phi''(\xi)\,(\tfrac{1}{2}h)^2+\frac{P_2}{4!}\phi^{\mathrm{iv}}(\xi)\,(\tfrac{1}{2}h)^4-\ldots\right\}$$

$$+h\left\{\phi_2(\xi+\tfrac{1}{2}h)-\phi_2(\xi-\tfrac{1}{2}h)\right\}, \tag{14}$$

where

$$\phi_2(\xi)=\tfrac{1}{2}\left\{\frac{1}{D_1+D_2}-\frac{D_2\sinh D_1}{D_1\sinh D_2\sinh(D_1+D_2)}\right\}f(\xi)\,\phi(\xi)$$

$$=\tfrac{1}{2}\frac{D_2}{D_1}\left\{\left(\coth\frac{D_1+D_2}{2}-\frac{1}{D_1+D_2}\right)-\left(\coth\frac{D_2}{2}-\frac{1}{D_2}\right)\right\}f(\xi)\,\phi(\xi).$$

Since

$$\tfrac{1}{2}\theta\coth\tfrac{1}{2}\theta=1+\frac{B_1}{2!}\theta^2-\frac{B_2}{4!}\theta^4+\ldots, \tag{15}$$

this reduces to

$$\phi_2(\xi)=F(\xi)\left\{\frac{B_1}{2!}h\frac{d}{d\xi}-\frac{B_2}{4!}h^3\frac{d^3}{d\xi^3}+\ldots\right\}\phi'(\xi)$$

$$-\left\{\frac{B_1}{2!}h\frac{d}{d\xi}-\frac{B_2}{4!}h^3\frac{d^3}{d\xi^3}+\ldots\right\}F(\xi)\,\phi'(\xi). \tag{16}$$

Substituting in (14), writing $\xi=\xi_1,\,\xi_2,\,\ldots,\,\xi_p$ successively, and adding

together the p equations so obtained, we have, finally,

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=1}^{r=p} A_r \left\{ \phi(\xi_r) - \frac{P_1}{2!}(\tfrac{1}{2}h)^2 \phi''(\xi_r) + \frac{P_2}{4!}(\tfrac{1}{2}h)^4 \phi^{\text{iv}}(\xi_r) - \dots \right\}$$

$$+ h\left[F(x) \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} \phi'(x) \right.$$

$$\left. - \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} F(x)\phi'(x) \right]_{x=x_0}^{x=x_p}. \quad (17)$$

This result can be verified by comparing it with (5B) and (5C). To give the expression in square brackets explicitly in terms of $f(x)$ and $\phi(x)$ and their derived functions, we write, for convenience,

$$Q_r = \frac{B_r}{2r}, \quad (18)$$

and we find that

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=1}^{r=p} A_r \left\{ \phi(\xi_r) - \frac{P_1}{2!}(\tfrac{1}{2}h)^2 \phi''(\xi_r) + \frac{P_2}{4!}(\tfrac{1}{2}h)^4 \phi^{\text{iv}}(\xi_r) - \dots \right\}$$

$$+ h^2 \left[\left\{ \frac{Q_1}{0!}\phi'(x) - \frac{Q_2}{2!}h^2\phi'''(x) + \frac{Q_3}{4!}h^4\phi^{\text{v}}(x) - \dots \right\} f(x) \right.$$

$$- \frac{1}{2!}\left\{ \frac{Q_2}{1!}\phi''(x) - \frac{Q_3}{3!}h^2\phi^{\text{iv}}(x) + \frac{Q_4}{5!}h^4\phi^{\text{vi}}(x) - \dots \right\} h^2 f'(x)$$

$$- \frac{1}{3!}\left\{ \frac{Q_2}{0!}\phi'(x) - \frac{Q_3}{2!}h^2\phi'''(x) + \frac{Q_4}{4!}h^4\phi^{\text{v}}(x) - \dots \right\} h^2 f''(x)$$

$$+ \frac{1}{4!}\left\{ \frac{Q_3}{1!}\phi''(x) - \frac{Q_4}{3!}h^2\phi^{\text{iv}}(x) + \frac{Q_5}{5!}h^4\phi^{\text{vi}}(x) - \dots \right\} h^4 f'''(x)$$

$$\left. + \&c. \right]_{x=x_0}^{x=x_p}, \quad (19)$$

where

$$\left.\begin{array}{llllll} P_1 = \frac{1}{3}, & P_2 = \frac{7}{15}, & P_3 = \frac{31}{21}, & P_4 = \frac{127}{15}, & P_5 = \frac{2555}{33}, & \dots \\ Q_1 = \frac{1}{12}, & Q_2 = \frac{1}{120}, & Q_3 = \frac{1}{252}, & Q_4 = \frac{1}{240}, & Q_5 = \frac{1}{1332}, & \dots \end{array}\right\}. \quad (20)$$

5. The value of $\phi_1(\xi)$ might have been obtained without using the symbolic method, as follows.

Let us assume that

$$\phi_1(\xi) = \phi(\xi) + \frac{\kappa_1}{2!}(\tfrac{1}{2}h)^2\phi''(\xi) + \frac{\kappa_2}{4!}(\tfrac{1}{2}h)^4\phi^{\text{iv}}(\xi) + \dots .$$

(The odd differential coefficients are omitted, since it is clear that $\phi_1(\xi)$ is not altered by changing $\frac{1}{2}h$ into $-\frac{1}{2}h$). Expand $f(\xi+\theta)$ and $\phi(\xi+\theta)$ in ascending powers of θ by Taylor's theorem, substitute in (6), and perform the integrations; then substitute for $\phi_1(\xi)$ from the above expression, and arrange the result in a series proceeding by ascending powers of h. The successive coefficients will contain products as follows:—

co. h^3	co. h^5	co. h^7	
$f(\xi)\,\phi''(\xi)$,	$f(\xi)\,\phi^{\text{iv}}(\xi)$,	$f(\xi)\,\phi^{\text{vi}}(\xi)$,	...,
$f'(\xi)\,\phi'(\xi)$,	$f'(\xi)\,\phi'''(\xi)$,	$f'(\xi)\,\phi^{\text{v}}(\xi)$,	...,
	$f''(\xi)\,\phi''(\xi)$,	$f''(\xi)\,\phi^{\text{iv}}(\xi)$,	...,
	$f'''(\xi)\,\phi'(\xi)$,	$f'''(\xi)\,\phi'''(\xi)$,	...,
		$f^{\text{iv}}(\xi)\,\phi''(\xi)$,	...,
		$f^{\text{v}}(\xi)\,\phi'(\xi)$,	...,
			... .

The first coefficient will involve κ_1, the second κ_1 and κ_2, and so on. Hence the values of $\kappa_1, \kappa_2, \dots$ can be chosen successively so that each term in the expansion of $\Phi(\xi)\,d\xi$ in ascending powers of h shall be an exact differential. The n^{th} term, divided by $\frac{2}{2n+1}(\frac{1}{2}h)^{2n+1}$, is

$$\begin{aligned}\Big\{ &\frac{1}{0!}\,\frac{1}{(2n)!}\Big(1-\frac{2n+1}{1}\kappa_n\Big) f(\xi)\,\phi^{(2n)}(\xi) + \frac{1}{1!}\,\frac{1}{(2n-1)!} f'(\xi)\,\phi^{(2n-1)}(\xi) \\ &+ \frac{1}{2!}\,\frac{1}{(2n-2)!}\Big(1-\frac{2n+1}{3}\kappa_{n-1}\Big) f''(\xi)\,\phi^{(2n-2)}(\xi) + \dots \\ \dots &+ \frac{1}{(2n-2)!}\,\frac{1}{2!}\Big(1-\frac{2n+1}{2n-1}\kappa_1\Big) f^{(2n-2)}(\xi)\,\phi'(\xi) \\ &+ \frac{1}{(2n-1)!}\,\frac{1}{1!} f^{(2n-1)}(\xi)\,\phi'(\xi)\Big\}\,d\xi.\end{aligned}$$

Let us choose κ_n so as to make this the differential of

$$C_1\frac{1}{1!}\,\frac{1}{(2n-1)!} f(\xi)\,\phi^{(2n-1)}(\xi) + C_2\frac{1}{2!}\,\frac{1}{(2n-2)!} f'(\xi)\,\phi^{(2n-2)}(\xi) + \dots$$

$$\dots + C_{2n-1}\frac{1}{(2n-1)!}\,\frac{1}{1!} f^{(2n-2)}(\xi)\,\phi'(\xi).$$

Differentiating this last expression, and equating coefficients,

$$\begin{aligned}
1-\frac{2n+1}{1}\kappa_n &= \frac{2n}{1}C_1,\\
1 &= C_1 + \frac{2n-1}{2}C_2,\\
1-\frac{2n+1}{3}\kappa_{n-1} &= C_2 + \frac{2n-2}{3}C_3,\\
\cdots \quad \cdots \quad &\cdots \quad \cdots \quad \cdots\\
1-\frac{2n+1}{2n-1}\kappa_1 &= C_{2n-2} + \frac{2}{2n-1}C_{2n-1},\\
1 &= C_{2n-1}.
\end{aligned}$$

Eliminating, and remembering that

$$1-\frac{2n}{1}+\frac{2n\,(2n-1)}{1.2}-\ldots+\frac{2n\,(2n-1)\ldots 1}{1.2\ldots 2n}=(1-1)^{2n}=0,$$

we find that

$$\frac{1}{0!\,(2n+1)!}+\frac{\kappa_1}{2!\,(2n-1)!}+\frac{\kappa_2}{4!\,(2n-3)!}+\ldots+\frac{\kappa_n}{(2n)!\,1!}=0.$$

Now let $$\Omega(\theta)=1+\frac{\kappa_1}{2!}\theta^2+\frac{\kappa_2}{4!}\theta^4+\ldots.$$

Then the above relation shows that

$$\Omega(\theta).\sinh\theta=\theta,$$

and therefore $$\Omega(\theta)=\frac{\theta}{\sinh\theta},$$

which agrees with (14).

6. In the general formula (17) or (19) the expressions to be summed are given in terms of $\xi_1, \xi_2, \ldots, \xi_p$, the abscissæ of the middle points of the bases of the compartments $A_1, A_2, \ldots, A_p$. If preferred, we can use $x_0, x_1, \ldots, x_p$. For

$$\begin{aligned}
\phi_1(\xi) &= \frac{\frac{1}{2}h\frac{d}{d\xi}}{\sinh\frac{1}{2}h\frac{d}{d\xi}}\phi(\xi)\\
&= \frac{h\frac{d}{d\xi}}{\sinh h\frac{d}{d\xi}}\tfrac{1}{2}\left\{\phi(\xi+\tfrac{1}{2}h)+\phi(\xi-\tfrac{1}{2}h)\right\}, \qquad (21)
\end{aligned}$$

and therefore

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=0}^{r=p} \tfrac{1}{2}(A_r + A_{r+1}) \left\{ \phi(x_r) - \frac{P_1}{2!} h^2 \phi''(x_r) + \frac{P_2}{4!} h^4 \phi^{\text{iv}}(x_r) - \dots \right\}$$

$$+ h \left[F(x) \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} \phi'(x) \right.$$

$$\left. - \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} F(x)\,\phi'(x) \right]_{x=x_0}^{x=x_p}, \qquad (22)$$

where A_0 and A_{p+1} are each zero.

II. Application to Calculation of Σx^m.*

7. When an attribute common to a number of individuals is capable of exact measurement, and of representation by a linear magnitude x, whose variation is continuous, the result of a classification of the individuals according to this attribute is generally given in the form of a table showing the numbers which fall into consecutive classes corresponding to equal increments of x, *e.g.*,

(A) *Weights of* 5,552 *Englishmen.*†

Weight.	Number.	Weight.	Number.
90 lbs. to 100 lbs.	2	180 lbs. to 190 lbs.	304
100 lbs. to 110 lbs.	26	190 lbs. to 200 lbs.	174
110 lbs. to 120 lbs.	133	200 lbs. to 210 lbs.	75
120 lbs. to 130 lbs.	338	210 lbs. to 220 lbs.	62
130 lbs. to 140 lbs.	694	220 lbs. to 230 lbs.	33
140 lbs. to 150 lbs.	1,240	230 lbs. to 240 lbs.	10
150 lbs. to 160 lbs.	1,075	240 lbs. to 250 lbs.	9
160 lbs. to 170 lbs.	881	250 lbs. to 260 lbs.	3
170 lbs. to 180 lbs.	492	260 lbs. to 270 lbs.	1

* Some of the results of this section have been stated, without proof, in a paper printed in the *Journal of the Royal Statistical Society* (September, 1897, p. 698).

† *Report of the British Association*, 1883, p. 257.

(B) *Examination Marks of* 1,000 *Sandhurst Candidates.* *

Mark.	Number.	Mark.	Number.
0 to 99	15	500 to 599	221
100 to 199	54	600 to 699	115
200 to 299	131	700 to 799	31
300 to 399	227	800 to 899	3
400 to 499	203	900 to 999	0

From the statistical point of view we are not concerned with particular individuals. We regard them—on grounds which need not be discussed here—as having been obtained by random selection from an indefinitely great community; and we investigate the distribution of the values of x in this hypothetical community. Any such distribution can be expressed in terms of the mean values of $x, x^2, x^3, \ldots$ for the community, which we may denote by $[x], [x^2], [x^3], \ldots$; and therefore the problem is to determine, from the data, the most probable combination of values of $[x], [x^2], [x^3], \ldots$.

To do this, when the data take the form of a table such as (A) or (B), we adopt the same process as if we knew the exact value of x for each of the observed individuals. We postulate an indefinite number of communities with different laws of distribution of x, and therefore with different values of $[x], [x^2], [x^3], \ldots$; we suppose that from each of these communities a random selection of n individuals is made N times, where n is the number of individuals actually observed, and N is an indefinitely great number; and from this double aggregate of sets of n individuals we pick out those sets in which the numbers in the consecutive classes are the same as in the actual data, and consider how the different values of $[x], [x^2], [x^3], \ldots$ are distributed in these particular cases.

8. Let the range of values of x, in the original community, be from x_0 to x_p, this range being divided into p segments, each equal to h; and let the actual data consist of a table giving numbers $n_1, n_2, \ldots n_p$ in the corresponding p classes, where $n_1 + n_2 + \ldots + n_p = n$.

* These are the marks in English Composition (a few candidates being omitted) at the examinations in November, 1894, and June, 1895. The table must not be taken as giving the law of distribution of the marks at either examination separately.

We assume that the data show that both the variation of x and the frequency of the variation may be regarded as continuous; *i.e.*, that the true figure of frequency of x may be regarded as bounded by a continuous curve. Let the equation to this unknown curve be

$$z = f(x),$$

and let the areas between the successive ordinates $f(x_0), f(x_1), \ldots, f(x_p)$, where $x_r = x_0 + xh$, be $A_1, A_2, \ldots, A_p$, the total area of the curve being A. Then our fundamental postulate is that there are an indefinite number of forms of $f(x)$, or, rather, of $f(x) \div A$; and we have to consider the result of a random selection of n individuals when $f(x) \div A$ has any particular form, and thence to determine the distribution of the different values of

$$[x^m] \equiv \int_{x_0}^{x_p} f(x)\, x^m\, dx \div A,$$

when the random selection gives numbers $n_1, n_2, \ldots, n_p$ in the p classes.

If in the expression contained in square brackets in (17) we replace the differential coefficients by differences, and if the resulting series is initially convergent when

$$\phi(x) = x^m,$$

we obtain $\int_{x_0}^{x_p} f(x)\, x^m\, dx$ in the form

$$C_1 A_1 + C_2 A_2 + \ldots + C_p A_p.$$

Let

$$n'_r = nA_r/A,$$

so that $n'_1, n'_2, \ldots, n'_p$ are the numbers (not necessarily integral) that would fall into the p classes if we made a *representative* selection of n individuals. Then

$$n[x^m] \equiv n \int_{x_0}^{x_p} f(x)\, x^m\, dx \div A$$

is of the form

$$C_1 n'_1 + C_2 n'_2 + \ldots + C_p n'_p.$$

By hypothesis, the numbers $n_1, n_2, \ldots, n_p$ are obtained by *random* selection, and they therefore differ from $n'_1, n'_2, \ldots, n'_p$ by amounts which, when n is large, are distributed according to the normal law with mean values zero, and mean squares $n'_1(n-n'_1)/n$, $n'_2(n-n'_2)/n, \ldots, n'_p(n-n'_p)/n$,

the p distributions being normally correlated. It follows, by the ordinary methods of inverse probabability, that, if the *a priori* frequency of different values of $[x^m]$, in those cases in which $A_1, A_2, \ldots, A_p$ are nearly proportional to $n_1, n_2, \ldots n_p$, is continuous, the most probable value of $n[x^m]$ as deducible from the data is (approximately) $C_1n_1 + C_2n_2 + \ldots + C_pn_p$; and the possible values are distributed normally about this value with mean square $\{\Sigma C_r^2 n_r - (\Sigma C_r n_r)^2\} \div n$. Similarly, it may be shown that the different distributions obtained by giving different values to m are normally correlated; and it follows that the most probable combination of values of $[x], [x^2], [x^3], \ldots$ is to be found by assuming the areas $A_1, A_2, \ldots, A_p$ of the figure of frequency to be proportional to $n_1, n_2, \ldots, n_p$. It is understood, throughout, that n is a large number.

Denote the most probable value of $[x^m]$ by "$[x^m]$". Then, writing $\phi(x) = x^m$ in (17), multiplying both sides of the equation by n/A, and replacing nA_r/A by n_r, we have

$$n\text{"}[x^m]\text{"} = \sum_{r=1}^{r=p} n_r \left(\xi_r^m - \tfrac{1}{4} P_1 \frac{m!}{2!(m-2)!} h^2 \xi_r^{m-2} + \tfrac{1}{16} P_2 \frac{m!}{4!(m-4)!} h^4 \xi_r^{m-4} - \ldots \right)$$

$$- n \left\{ B_1 \frac{m!}{2!\,(m-2)!} h^2 x_0^{m-2} - B_2 \frac{m!}{4!\,(m-4)!} h^4 x_0^{m-4} + \ldots \right\}$$

$$- \frac{n}{A} \left[\left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \ldots \right\} mhx^{m-1} \int_x^{x_p} f(x)\, dx \right]_{x=x_0}^{x=x_p}, \quad (23)$$

where $$\xi_r = \tfrac{1}{2}(x_{r-1} + x_r).$$

To express this in a convenient form, let M_q denote the *rough* value of Σx^q for the n individuals, obtained by massing the individuals in each class at the centre of the range of the class; *i.e.*, by taking every value of x between x_{r-1} and x_r as equal to the arithmetic mean of x_{r-1} and x_r. Thus

$$M_q = \sum_{r=1}^{r=p} n_r \xi_r^q.$$

Also let $$n_r + n_{r+1} + \ldots + n_p = N_{r-1},$$

so that $$N_0 = n, \quad N_p = 0;$$

and let $$m(N_0 x_0^{m-1} + N_p x_p^{m-1}) = S_0, \quad m(N_1 x_1^{m-1} + N_{p-1} x_{p-1}^{m-1}) = S_1,$$

$$m(N_2 x_2^{m-1} + N_{p-2} x_{p-2}^{m-1}) = S_2, \ \ldots .$$

Then, if $S_1 - S_0 = \Delta S_0$, $S_2 - S_1 = \Delta S_1$, ..., $\Delta S_1 - \Delta S_0 = \Delta^2 S_0$, ..., (23) becomes

$$n\,"[x^m]" = M_m - \tfrac{1}{4}P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{16}P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \ldots$$
$$- n \left\{ \tfrac{1}{6} \frac{m!}{2!\,(m-2)!} h^2 x_0^{m-2} - \tfrac{1}{30} \frac{m!}{4!\,(m-4)!} h^4 x_0^{m-4} + \ldots \right\}$$
$$+ \tfrac{1}{12} h \left\{ \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{60}\Delta^3 - \tfrac{9}{40}\Delta^4 + \tfrac{863}{60480}\Delta^5 - \ldots \right\} S_0. \qquad (24)$$

9. The data may be studied by another method, which avoids the difficulties incident to inverse probability, and is therefore incomplete. If we knew the exact value of x for each of the n individuals, we should calculate the values of $\Sigma x \div n$, $\Sigma x^2 \div n$, $\Sigma x^3 \div n$, ...; and these calculated averages would constitute the data for determining the mean values $[x]$, $[x^2]$, $[x^3]$, ... When we only know the numbers $n_1, n_2, \ldots, n_p$ in the p classes, we cannot calculate the value of Σx^m exactly. We may, therefore, leave on one side the question of the amount of information that the value of Σx^m gives as to the frequency-constants of the original community, and consider only what is the most probable value of Σx^m itself, on the assumption that the numbers $n_1, n_2, \ldots, n_p$ are obtained by random selection from some particular (unknown) community, and how, on the same assumption, the different possible values are distributed about this most probable value.

Let
$$z = f(x)$$
be the equation to the curve of frequency in the original community; and, as before, let
$$n_r' = nA_r/A.$$

Since the numbers $n_1, n_2, \ldots, n_p$ are obtained by random selection, the difference $n_r \sim n_r'$ may be regarded as comparable with $\sqrt{n_r(n-n_r)/n}$. To examine the distribution of possible values of Σx^m, we suppose an indefinitely great number of sets of n individuals to be obtained by random selection, and we pick out the cases in which the numbers in the p classes are $n_1, n_2, \ldots, n_p$. Let $\Sigma_r x^m$ denote the portion of Σx^m which is due to the n_r individuals for which x lies between x_{r-1} and x_r. Then it is clear, from the principles of random selection, that the distributions of $\Sigma_1 x^m$, $\Sigma_2 x^m$, ..., $\Sigma_p x^m$, and of their sum Σx^m, in the special cases considered, are the same as if we made *independent* random selections of $n_1, n_2, \ldots, n_p$ individuals from the p classes of the original community.

Let $(x^m)_r$ denote the mean value of x^m for those individuals in the original community for which x lies between x_{r-1} and x_r, so that

$$(x^m)_r = \int_{x_{r-1}}^{x_r} f(x)\, x^m dx \div \int_{r-1}^{x_r} f(x)\, dx = \int_{x_{r-1}}^{x_r} f(x)\, x^m dx \div A_r.$$

Then the theory of error shows that, if n_r individuals were taken at random from this particular class, the mean value of $\Sigma_r x^m$ would be $n_r (x^m)_r$, and the different values would be distributed about this mean with mean square $n_r\left[(x^{2m})_r - \{(x^m)_r\}^2\right]$. It follows from the last paragraph that the different values of Σx^m are distributed about a mean value $\sum_{r=1}^{r=p} n_r (x^m)_r$ with mean square $\sum_{r=1}^{r=p} n_r \left[(x^{2m})_r - \{(x^m)_r\}^2\right]$. This is true whatever the value of n may be. When n is large, the distribution is approximately normal, and the most probable value of Σx^m is the same as the mean value $\sum_{r=1}^{r=p} n_r (x^m)_r$, while the actual value may be regarded as differing from this by an amount comparable with

$$\sqrt{\sum_{r=1}^{r=p} n_r \left[(x^{2m})_r - \{(x^m)_r\}^2\right]},$$

i.e., by an amount comparable with

$$\sqrt{n}\sqrt{\frac{\Sigma x^{2m}}{n} - \sum_{r=1}^{r=p} \frac{n'_r}{n} \{(x^m)_r\}^2}.$$

Let "Σx^m" denote the most probable value. Then, from (14), with the same notation as before,

$$\begin{aligned}
\text{"}\Sigma x^m\text{"} &= \sum_{r=1}^{r=p} n_r (x^m)_r \\
&= \sum_{r=1}^{r=p} \frac{n_r}{A_r} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r + \theta)(\xi_r + \theta)^m d\theta \\
&= \sum_{r=1}^{r=p} n_r \left\{ \xi_r^m - \tfrac{1}{4} P_1 \frac{m!}{2!(m-2)!} h^2 \xi_r^{m-2} + \tfrac{1}{16} P_2 \frac{m!}{4!(m-4)!} h^4 \xi_r^{m-4} - \ldots \right\} \\
&\quad + \sum_{r=1}^{r=p} \frac{n_r}{A_r} \left[h\phi_2(x) \right]_{x=x_{r-1}}^{x=x_r} \\
&= M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{16} P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \ldots \\
&\quad + \frac{n}{A} \left[h\phi_2(x) \right]_{x=x_0}^{x=x_p} \\
&\quad + \sum_{r=1}^{r=p} \frac{n_r - n'_r}{A_r} \left[h\phi_2(x) \right]_{x=x_{r-1}}^{x=x_r},
\end{aligned} \qquad \ldots \quad (25)$$

where

$$h\phi_2(x) = \left\{ B_1 \frac{m!}{2!\,(m-2)!} h^2 x^{m-2} - B_2 \frac{m!}{4!\,(m-4)!} h^4 x^{m-4} + \dots \right\} \int_x^{x_p} f(x)\,dx$$

$$- \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} mhx^{m-1} \int_x^{x_p} f(x)\,dx. \quad (26)$$

The value of "Σx^m", as given by (25), and the value of n"$[x^m]$", as given by (23) or (24), agree in the terms

$$M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{16} P_2 \frac{m!}{4!\,(m-4)!} M_{m-2} h^4 - \dots,$$

but the remaining terms are different. The difference, however, is a linear function of the differences $n_1 - n_1'$, $n_2 - n_2'$, ..., $n_p - n_p'$; and these are small in comparison with n. Hence n"$[x^m]$" and "Σx^m" may be regarded as substantially the same, the value of either being given by (24).

10. The formula (24) applies to any distribution which satisfies the condition as to continuity. In the special cases we are considering, the extreme values $n_0, n_1, \dots$, and $n_p, n_{p-1}, \dots$ are very small. We have then, approximately,

$$n\text{"}[x^m]\text{"} = \text{"}\Sigma x^m\text{"}$$
$$= M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{16} P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \dots, \quad (27)$$

which gives, as particular cases,

$$\left.\begin{aligned} n\text{"}[x]\text{"} &= \text{"}\Sigma x\text{"} = M_1 \\ n\text{"}[x^2]\text{"} &= \text{"}\Sigma x^2\text{"} = M_2 - \tfrac{1}{12} n h^2 \\ n\text{"}[x^3]\text{"} &= \text{"}\Sigma x^3\text{"} = M_3 - \tfrac{1}{4} M_1 h^2 \\ n\text{"}[x^4]\text{"} &= \text{"}\Sigma x^4\text{"} = M_4 - \tfrac{1}{2} M_2 h^2 + \tfrac{7}{240} n h^4 \\ n\text{"}[x^5]\text{"} &= \text{"}\Sigma x^5\text{"} = M_5 - \tfrac{5}{6} M_3 h^2 + \tfrac{7}{48} M_1 h^4 \\ &\&c. \end{aligned}\right\}, \quad (28)$$

since, of course, $M_0 = n$.

This is the most convenient form when we are able to take the zero at the middle point of one of the divisions of the range, so that ξ may be integral multiples of h. In some cases it may be best to take a point of division, so that $x_0, x_1, x_2, \dots, x_p$

will be integral multiples of h. Let M_1', M_2', ..., M_m' denote the rough values of Σx, Σx^2, ..., Σx^m obtained by massing $\frac{1}{2}n_1$ individuals at x_0, $\frac{1}{2}(n_1+n_2)$ at x_1, $\frac{1}{2}(n_2+n_3)$ at x_2, ..., $\frac{1}{2}n_p$ at x_p. Then, by (22),

$$\left.\begin{aligned} n\text{“}[x]\text{”} &= \text{“}\Sigma x\text{”} &&= M_1' \\ n\text{“}[x^2]\text{”} &= \text{“}\Sigma x^2\text{”} &&= M_2' - \tfrac{1}{3}nh^2 \\ n\text{“}[x^3]\text{”} &= \text{“}\Sigma x^3\text{”} &&= M_3' - M_1'h^2 \\ n\text{“}[x^4]\text{”} &= \text{“}\Sigma x^4\text{”} &&= M_4' - 2M_2'h^2 + \tfrac{7}{15}nh^4 \\ n\text{“}[x^5]\text{”} &= \text{“}\Sigma x^5\text{”} &&= M_5' - \tfrac{10}{3}M_3'h^2 + \tfrac{1}{3}M_1'h^4 \\ & & &\&c. \end{aligned}\right\}. \qquad (29)$$

If we use the rough values M_1, M_2, ... to determine μ_2, μ_3, ..., the mean square, mean cube, ... of the deviation from the mean, and if the common difference h is the unit of measurement, we have, from (28), μ_1 being zero,

$$\left.\begin{aligned} \text{corrected value of } \mu_2 &= \mu_2 - \tfrac{1}{12} \\ &= \mu_2 - \cdot 083 \\ \text{corrected value of } \mu_4 &= \mu_4 - \tfrac{1}{2}\mu_2 + \tfrac{7}{240} \\ &= \mu_4 - \tfrac{1}{2}\mu_2 + \cdot 0291\dot{6} \\ \text{corrected value of } \mu_5 &= \mu_5 - \tfrac{5}{6}\mu_3 \end{aligned}\right\}, \qquad (30)$$

the value of μ_3 being unaltered. Similarly, if μ_2', μ_3', ... denote the values of μ_2, μ_3, ... as deduced from M_1', M_2', M_3', ..., we have, from (29),

$$\left.\begin{aligned} \text{corrected value of } \mu_2 &= \mu_2' - \cdot\dot{3} \\ \text{corrected value of } \mu_4 &= \mu_4' - 2\mu_1' + \cdot 4\dot{6} \\ \text{corrected value of } \mu_5 &= \mu_5' - \tfrac{10}{3}\mu_3' \end{aligned}\right\}. \qquad (31)$$

11. The following are some numerical examples* showing the extent of the inaccuracy due to determining “Σx”, “Σx^2”, ... by the approximate formulæ (28), instead of by the accurate formula (25).

In the first example the selection is “representative,” the numbers being exactly proportional to the corresponding areas of the figure of frequency. The “most probable” values of Σx, Σx^2, ..., and of

* The first two of these illustrations are given in the paper referred to in note (*) on p. 362. A small error in one of the values has been corrected.

$\mu_3, \mu_5, \ldots$ are in this case obtained by a simple integration, and the differences between these values and the values given by (28) or (30) are due solely to the omitted term

$$\frac{n}{A}\left[h\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

which is equal to the second and third terms in (24). In the second example, taken from Kramp's table of values of $\int_t^\infty e^{-t^2}dt$, the selection is approximately representative, the integer nearest to n_r' being taken in each case. The most probable values are found by calculating each of the mean values $(x^m)_r$ separately, and taking the sum $\Sigma n_r (x^m)_r$; the differences between these and the values given by (28) or (30) are due principally to the term

$$\frac{n}{A}\left[h\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

but partly also to the fractional differences $n_r - n_r'$ occurring in the last term of (25). The third example is formed from the second by adding to or subtracting from the number in each class a number comparable with the "probable error" of the total number in the class. It may, therefore, be taken as a typical illustration of the amount of error that would occur in an actual case of random selection. The values of μ_2, μ_4, and μ_6 for the community selected being 2, 12, and 1680, the "probable errors" in μ_2 and μ_4 for a random selection of 88,622 individuals would be ·0064 and ·0888 respectively. The actual "errors" for the selected values of $n_1, n_2, \ldots, n_p$ are ·0156 and ·1815, so that the example is really an unfavourable one; but the formulæ in (30) give the values of μ_2 and μ_4 within ·0014 and ·0346 respectively, which is as close an approximation as we could expect to get.

In each case the "rough" values of $M_1, M_2, \ldots$ and the corresponding values of μ_2, μ_3, μ_4 are given for comparison, so as to show the degree of inaccuracy in the even moments μ_2, μ_4 which would be caused by omitting the corrections afforded by (28) or (30).

(i.) *Curve of Frequency* $z \propto (x+4\frac{1}{2})^5 (5\frac{1}{2}-x)^6$.

$n = 10^{12}$.

(Representative Selection.)

$x = -4\frac{1}{2}$	to	$-3\frac{1}{2}$	 541,231,822
$-3\frac{1}{2}$	to	$-2\frac{1}{2}$	 18,864,047,410
$-2\frac{1}{2}$	to	$-1\frac{1}{2}$	 98,443,459,630
$-1\frac{1}{2}$	to	$-\frac{1}{2}$	 216,942,703,570
$-\frac{1}{2}$	to	$\frac{1}{2}$	 278,001,526,318
$\frac{1}{2}$	to	$1\frac{1}{2}$	 228,994,738,642
$1\frac{1}{2}$	to	$2\frac{1}{2}$	 119,611,449,550
$2\frac{1}{2}$	to	$3\frac{1}{2}$	 34,697,711,410
$3\frac{1}{2}$	to	$4\frac{1}{2}$	 3,852,951,310
$4\frac{1}{2}$	to	$5\frac{1}{2}$	 50,180,338
Total	...	...	1,000,000,000,000

	$\Sigma x \div 10^6$	$\Sigma x^2 \div 10^6$	$\Sigma x^3 \div 10^6$	$\Sigma x^4 \div 10^6$	$\Sigma x^5 \div 10^6$
Rough value	115387	1871774	827127	9429592	8085022
Corrected value	115387	1788441	798281	8522872	7412577
Most probable value ...	115385	1788462	798077	8524038	7405402

	μ_2	μ_3	μ_4
Rough value	1·85847	·18227	9·19204
Corrected value.....	1·77513	·18227	8·29197
Most probable value...	1·77515	·18207	8·29803

(ii.) *Curve of Frequency* $z \propto e^{-\frac{1}{2}(x-2)^2}$.

$n = 88{,}622$.

(Approximately Representative Selection.)

$x = -6\frac{1}{2}$	to	$-5\frac{1}{2}$	 2	$x = \frac{1}{2}$	to	$1\frac{1}{2}$	 21,005
$-5\frac{1}{2}$	to	$-4\frac{1}{2}$	 37	$1\frac{1}{2}$	to	$2\frac{1}{2}$	 11,259
$-4\frac{1}{2}$	to	$-3\frac{1}{2}$	 355	$2\frac{1}{2}$	to	$3\frac{1}{2}$	 3,733
$-3\frac{1}{2}$	to	$-2\frac{1}{2}$	 2,098	$3\frac{1}{2}$	to	$4\frac{1}{2}$	 765
$-2\frac{1}{2}$	to	$-1\frac{1}{2}$	 7,670	$4\frac{1}{2}$	to	$5\frac{1}{2}$	 97
$-1\frac{1}{2}$	to	$-\frac{1}{2}$	 17,338	$5\frac{1}{2}$	to	$6\frac{1}{2}$	 8
$-\frac{1}{2}$	to	$\frac{1}{2}$	 24,255	Total	...		88,622

	Σx	Σx^2	Σx^3	Σx^4	Σx^5
Rough value	17726	188168	111560	1196948	1169816
Corrected value.....	17726	180783	107128½	1105449	1079434
Most probable value.....	17726	180783	107123	1105535	1079202

	μ_2	μ_3	μ_4
Rough value	2·08325	·00074	13·00394
Corrected value.....	1·99992	·00074	11·99148
Most probable value...	1·99992	·00068	11·99250

(iii.) *Curve of Frequency* $z \propto e^{-\frac{1}{2}(x-2)^2}$.

$n = 88{,}622.$

(Random Selection.)

$x = -6\frac{1}{2}$ to $-5\frac{1}{2}$	3	$x = \frac{1}{2}$ to $1\frac{1}{2}$	20,965
$-5\frac{1}{2}$ to $-4\frac{1}{2}$	31	$1\frac{1}{2}$ to $2\frac{1}{2}$	11,289
$-4\frac{1}{2}$ to $-3\frac{1}{2}$	374	$2\frac{1}{2}$ to $3\frac{1}{2}$	3,783
$-3\frac{1}{2}$ to $-2\frac{1}{2}$	2,153	$3\frac{1}{2}$ to $4\frac{1}{2}$	773
$-2\frac{1}{2}$ to $-1\frac{1}{2}$	7,657	$4\frac{1}{2}$ to $5\frac{1}{2}$	102
$-1\frac{1}{2}$ to $-\frac{1}{2}$	17,326	$5\frac{1}{2}$ to $6\frac{1}{2}$	12
$-\frac{1}{2}$ to $\frac{1}{2}$	24,154	Total ...	88,622

	Σx	Σx^2	Σx^3	Σx^4	Σx^5
Rough value	17798	189716	113060	1218440	1216388
Corrected value.....	17798	182331	108610½	1126167	1124767
Most probable value.....	17795	182202	108288	1122875	1116061

	μ_2	μ_3	μ_4
Rough value	2·10040	·00198	13·23706
Corrected value	2·01707	·00198	12·21603
Most probable value ...	2·01563	−·00038	12·18146

III. Extension to Two or More Variables.

12. The method of § 4 can be extended to any number of variables. The expression

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}\cdot\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}\cdots-\frac{\sinh D_1}{D_1}\cdot\frac{D_2}{\sinh D_2}\cdot\frac{\sinh D_1'}{D_1'}\cdot\frac{D_2'}{\sinh D_2'}\cdots,$$

can be expanded so as to contain only positive integral powers (including the power 0) of $D_1, D_2, D_1', D_2', \ldots$; and it vanishes when $D_1+D_2, D_1'+D_2', \ldots$ are simultaneously equal to zero. Thus we see

that $$\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x, y, \ldots)\,\phi(x, y, \ldots)\,dx\,dy \ldots \qquad (32)$$

is equal to

$$\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x, y, \ldots)\,dx\,dy \ldots \left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\frac{P_2}{4!}(\tfrac{1}{2}h)^4\frac{d^4}{d\xi^4}-\ldots\right\}$$
$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\frac{P_2}{4!}(\tfrac{1}{2}k)^4\frac{d^4}{d\eta^4}-\ldots\right\}\ldots\phi(\xi,\eta,\ldots)+R, \qquad (33)$$

where R consists of terms involving the boundary values of $f(x, y, \ldots)$, $\phi(x, y, \ldots)$, and their partial differential coefficients with regard to $x, y, \ldots$. Hence, if the values of $x, y, \ldots$ range from $x = x_0$ to $x = x_0+ph$, from $y = y_0$ to $y = y_0+qk$, ..., and if $f(x, y, \ldots)$ and its first few differential coefficients are negligible for the extreme values of *each* of the variables $x, y, \ldots$, we have, approximately,

$$\Sigma\Sigma\ldots\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k}\ldots f(x, y, \ldots)\,\phi(x, y, \ldots)\,dx\,dy\ldots$$
$$=\Sigma\Sigma\ldots\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k}\ldots f(x, y, \ldots)\,dx\,dy\ldots$$
$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\ldots\right\}\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\ldots\right\}\ldots\phi(\xi,\eta,\ldots), \qquad (34)$$

the summations being made for the $p\times q\times\ldots$ terms corresponding to the different values of $\xi, \eta, \ldots$.

It will be sufficient to consider the case of two variables. Let D_1 and D_2 denote $\frac{1}{2}h\frac{d}{dx}$ operating on $f(x, y)$ and on $\phi(x, y)$ respectively, and let D_1' and D_2' have a similar meaning for $\frac{1}{2}k\frac{d}{dy}$. Then

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}\cdot\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}$$
$$=\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\cdot\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}$$
$$+\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\left\{\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}-\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}\right\}$$
$$+\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\right\}$$
$$+\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\right\}$$
$$\times\left\{\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}-\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}\right\}; \qquad (35)$$

and therefore $$\frac{1}{hk}\int_{x_0}^{x_p}\int_{y_0}^{y_q} f(x,y)\,\phi(x,y)\,dx\,dy$$

$$= \Sigma\Sigma \frac{1}{hk}\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dx\,dy \left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\} \times \left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(\xi,\eta)$$

$$+\Sigma\frac{1}{h}\left[\chi(D_1',D_2')\cdot\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h} f(x,y)\,dx\cdot\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}\phi(\xi,y)\right]_{y=y_0}^{y=y_q}$$

$$+\Sigma\frac{1}{k}\left[\chi(D_1,D_2)\cdot\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dy\cdot\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(x,\eta)\right]_{x=x_0}^{x=x_p}$$

$$+\Phi(x_p,y_q)-\Phi(x_0,y_q)-\Phi(x_p,y_0)+\Phi(x_0,y_0), \qquad (36)$$

where

$$\chi(D_1,D_2)\equiv\tfrac{1}{2}\frac{D_2}{D_1}\left\{\left(\coth\overline{D_1+D_2}-\frac{1}{D_1+D_2}\right)-\left(\coth D_2-\frac{1}{D_2}\right)\right\}, \qquad (37)$$

$$\Phi(x,y)\equiv\chi(D_1,D_2)\,\chi(D_1',D_2')\,f(x,y)\,\phi(x,y). \qquad (38)$$

The signs of summation in the first term denote, of course, that the expression is to be summed for each value of ξ with each value of η; while in the second the summation is for all values of ξ, and in the third it is for all values of η. These terms, which denote values along the edges of the area of integration, with the four concluding terms, which denote values at the corners, can be expanded as in (19); or they can be expressed in terms of differences, as in (5A) and (24). They are negligible when $f(x,y)$ and its first few differential coefficients $\frac{df}{dx}$, $\frac{df}{dy}$, ... vanish for all values of x when $y=y_0$ or y_q, and for all values of y when $x=x_0$ or x_p; and in these cases we have, approximately,

$$\int_{x_0}^{x_p}\int_{y_0}^{y_p} f(x,y)\,\phi(x,y)\,dx\,dy$$

$$= \Sigma\Sigma\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dx\,dy\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\} \times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(\xi,\eta). \qquad (39)$$

13. Now let the measurements of x and of y for n individuals be given by a table of double entry, showing for values of r from 1 to p, and for values of s from 1 to q, the number of individuals for which x lies between $x_0+(r-1)(x_p-x_0)/p$ and $x_0+r(x_p-x_0)/p$, and y between $y_0+(s-1)(y_q-y_0)/q$ and $y_0+s(y_q-y_0)/q$. Let "$\Sigma x^m y^t$" denote the most probable value of $\Sigma x^m y^t$ for the n individuals (or n times the most probable mean value of $x^m y^t$ for the original community), and let S_{mt} denote the rough value of $\Sigma x^m y^t$ obtained by taking the values of x and of y for all the individuals in each sub-class to be the arithmetic means of the values of x and of y respectively which determine the sub-class. Then, if $M_m (= S_{m0})$ and $N_t (= S_{0t})$ denote the "rough" values of Σx^m and of Σy^t obtained in this way, we have, approximately,

$$\left.\begin{aligned}
&\text{"}\Sigma xy\text{"} = S_{11}\text{*}\\
&\text{"}\Sigma x^2y\text{"} = S_{21}-\tfrac{1}{12}N_1h^2\\
&\text{"}\Sigma xy^2\text{"} = S_{12}-\tfrac{1}{12}M_1k^2\\
&\text{"}\Sigma x^3y\text{"} = S_{31}-\tfrac{1}{4}S_{11}h^2\\
&\text{"}\Sigma x^2y^2\text{"} = S_{22}-\tfrac{1}{12}M_2k^2-\tfrac{1}{12}N_2h^2+\tfrac{1}{144}nh^2k^2\\
&\text{"}\Sigma xy^3\text{"} = S_{13}-\tfrac{1}{4}S_{11}k^2\\
&\text{"}\Sigma x^4y\text{"} = S_{41}-\tfrac{1}{2}S_{21}h^2+\tfrac{7}{240}N_1h^4\\
&\text{"}\Sigma x^3y^2\text{"} = S_{32}-\tfrac{1}{4}S_{12}h^2-\tfrac{1}{12}M_2k^2+\tfrac{1}{48}M_1h^2k^2\\
&\text{"}\Sigma x^2y^3\text{"} = S_{23}-\tfrac{1}{4}S_{21}k^2-\tfrac{1}{12}N_2h^2+\tfrac{1}{48}N_1h^2k^2\\
&\text{"}\Sigma xy^4\text{"} = S_{14}-\tfrac{1}{2}S_{12}k^2+\tfrac{7}{240}M_1k^4\\
&\qquad\qquad \&c.
\end{aligned}\right\} \qquad (40),$$

IV. The Spurious Curve of Frequency.

14. Let $\frac{1}{h}\int_{x-\frac{1}{2}h}^{x+\frac{1}{2}h} f(x)\,dx$ be denoted by $f_1(x)$. If we call $z=f(x)$ the principal curve, then $z=f_1(x)$ may be called the derived curve.

* There is an apparent contradiction in this and similar formulæ when the value of y for each individual is equal or very nearly equal (or proportional) to that of x, *i.e.*, when the distributions of x and of y are very closely correlated; for the formula appears to give M_2 instead of $M_2-\frac{1}{12}h^2$. But, if y were always equal to x, there would be a discontinuity all along the line $y=x$; and therefore the formula, which is based on the hypothesis of continuity, would not apply at all. When y is only very nearly equal to x it will be seen that the sets of planes corresponding to consecutive values of x and y cut the solid of frequency in such a way that the volumes, which in forming the values of S_1 are concentrated at points outside the line $y=x$, are appreciable.

The curves are related in such a way that the ordinate to the latter curve at any point $x = \xi$ is proportional to the area of the former curve comprised between the ordinates at $x = \xi - \frac{1}{2}h$ and $x = \xi + \frac{1}{2}h$. If these ordinates are MP and $M'P'$, and if h is relatively small, the extremity of the ordinate to the derived curve will lie inside the area formed by the arc PP' and the chord PP'. Thus the derived curve lies inside the principal curve where the latter is convex, and outside it where it is concave; any point of intersection of the two curves being near a point of inflexion of the principal curve.

If $z = f(x)$ ranges from $x = x_0$ to $x = x_p$, then $z = f_1(x)$ only ranges from $x = x_0 + \frac{1}{2}h$ to $x = x_p - \frac{1}{2}h$. But, if $z = f(x)$ touches the base ($z = 0$) at its extremities, it may be considered to coincide with it beyond this range: and $z = f_1(x)$ may thus be regarded as extending from $x = x_0 - \frac{1}{2}h$ to $x = x_p + \frac{1}{2}h$; the curve, like the principal curve, touching the base at its extremities. We shall only consider this class of cases.

Let the areas of the principal curve between the successive ordinates corresponding to $x = x_0$, $x = x_0 + h$, $x = x_0 + 2h$, ..., $x = x_0 + ph$, be A_1, A_2, ..., A_p. Then the ordinates of the derived curve at the points $x = x_0 - \frac{1}{2}h$, $x = x_0 + \frac{1}{2}h$, ..., $x = x_p - \frac{1}{2}h$, $x = x_p + \frac{1}{2}h$ are $0, A_1/h, A_2/h, \ldots, A_p/h, 0$. Hence, from (2B), we have, approximately,

$$\text{area of derived curve} = \int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\,dx$$

$$= h\left\{\frac{A_1}{h} + \frac{A_2}{h} + \ldots + \frac{A_p}{h}\right\}$$

$$= \text{area of principal curve};$$

first moment of derived curve

$$= \int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\,x\,dx$$

$$= h\left\{\frac{A_1}{h}\xi_1 + \frac{A_2}{h}\xi_2 + \ldots + \frac{A_p}{h}\xi_p\right\}$$

$$= \text{first moment of principal curve, by (17).}$$

Thus the two curves have the same area and the same mean value of x. From (2B) we have as the general formula relating to the derived curve, when $A_1, A_2, \ldots, A_p$ are given,

$$\int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\,\phi(x)\,dx = \sum_{r=1}^{r=p} A_r \phi(x_r). \qquad (41)$$

15. When the result of a random selection of n individuals is to give numbers $n_1, n_2, \ldots, n_p$ in the successive classes corresponding to equidistant divisions of the scale of measurement, these numbers are, except for the "errors" due to random selection, proportional to the corresponding areas of the figure of frequency in the original community. In order, therefore, to represent the data graphically, we should draw a curve such that the areas between the successive ordinates at the $p+1$ points of the scale are proportional to $n_1, n_2, \ldots, n_p$. The calculation of the ordinates of this curve, which may be regarded as the true curve of frequency for the n individuals, involves troublesome interpolations; and, to avoid this, statisticians often draw a curve whose ordinates at the middle points of the divisions of the scale are proportional to $n_1, n_2, \ldots, n_p$. This curve may be called the *spurious curve of frequency*. It bears the same relation to the true curve of frequency that the derived curve of § 14 bears to the principal curve. Thus it has the same area as the true curve, and its central ordinate starts from the same point on the base, but it is flattened down so as to be inside the true curve at or about the mean and outside it at the extremities; and therefore it has a greater mean square of deviation.

If we deal with the spurious curve instead of with the true curve, we must apply the formula (41) instead of the corresponding formula derived from (17). Hence, in order to find the values of μ_2, μ_3, &c., for the spurious curve, we must use the "rough" values M_1, M_2, M_3, ..., without the introduction of the corrections given by (28) or (30).

The divergence of the spurious curve from the true curve is greater or less according as h is greater or less. The same facts may therefore be represented by any number of different spurious curves, according to the unit of measurement we adopt. Thus example (A) of § 7 would give as ordinates of the spurious curve, when the unit of measurement is 10 lbs. :—

Weight.	Ordinate.
115 lbs.	133
145 lbs.	1,240
175 lbs.	492
205 lbs.	75
235 lbs.	10

but, if the unit had been 30 lbs., the ordinates (reduced in th[illegible]

of 1 : 3) would have been

Weight.	Ordinate.
115 lbs.	166
145 lbs.	1,003
175 lbs.	559
205 lbs.	104
235 lbs.	17

which would have given a very different curve. Moreover, the results obtained by using the spurious curve are useless if we require an accurate test of any particular hypothesis. The arguments which lead us to expect to find a curve of a particular kind [*e.g.*, in Prof. Karl Pearson's recent researches, a curve whose equation is of the form $z \propto (x-x_0)^{\alpha}(x_p-x)^{\beta}$] relate to the curve of frequency itself, not to the spurious curve; and, by attempting to fit a curve of this kind to the spurious curve, we shall obtain incorrect results.

Appendix.

Moments of a Polygon.

The question of the proper formula to be used for Σx^m has been incidentally discussed by Prof. Karl Pearson in a recent memoir.* His method is to draw ordinates proportional to the numbers given by the observations, and to join the tops of these ordinates by straight lines, so as to form a polygon. It will be seen that this polygon lies inside the spurious curve where the latter is convex, and outside it where it is concave; and that, when $z = f(x)$ has close contact with the base at its extremities, the area of the polygon is approximately equal to that of the curve, and the first moments of the two areas are also approximately equal. The "mean square of deviation" of the polygon is, therefore, greater than that of the spurious curve, which, again, is greater than that of the true curve.

It is interesting, though the result is of no practical value, to obtain the moments of such a polygon in terms of the areas, instead of in terms of the ordinates. Let ordinates $z_0, z_1, \ldots, z_p$ be drawn at points $x = x_0$, $x = x_1$, ..., $x = x_p$, the common difference of abscissa being h, and let the tops of the ordinates be joined so as to form a polygon. This polygon consists of a series of trapezia of areas $A_1, A_2, \ldots, A_p$, where

$$A_r = \tfrac{1}{2}h\,(z_{r-1}+z_r).$$

* *Phil. Trans.*, Vol. CLXXXVI. (1895), A, p. 349.

The m^{th} moment of the polygon about $x = 0$ is

$$\sum_{r=1}^{r=p} \int_{x_{r-1}}^{x_r} zx^m\,dx.$$

Let this be denoted by P_m. Then, as Prof. Pearson shows,

$$\int_{x_{r-1}}^{x_r} zx^m\,dx$$

$$= -\frac{1}{(m+1)(m+2)}\,\frac{1}{h}\,(z_r - z_{r-1})(x_r^{m+2} - x_{r-1}^{m+2}) + \frac{1}{m+2}\,(z_r x_r^{m+1} - z_{r-1}x_{r-1}^{m+1}) \tag{42}$$

and this gives

$$\sum_{r=1}^{r=p} \int_{x_{r-1}}^{x_r} zx^m\,dx$$

$$= 2\sum_{r=0}^{r=p} \left\{ z_r \left(\frac{m!}{2!\,m!} x_r^m h + \frac{m!}{4!\,(m-2)!}\,x_r^{m-2}h^3 + \ldots \right) \right\}$$

$$- \left[z \left(\frac{m!}{3!\,(m-1)!}\,x^{m-1}h^2 + \frac{m!}{5!\,(m-3)!}\,x^{m-3}h^4 + \ldots \right) \right]_{x=x_0}^{x=x_p}, \tag{43}$$

the expression under the sign of summation being only taken once for the extreme values, instead of twice. The term given by the square brackets vanishes when z_0 and z_p are both zero.

To express P_m in terms of $A_1, A_2, \ldots, A_p$, we have, from (42),

$$\int_{x_{r-1}}^{x_r} zx^m\,dx = -\frac{1}{(m+1)(m+2)}\,\frac{2}{h}\,(z_r x_r^{m+2} + z_{r-1}x_{r-1}^{m+2})$$

$$+ \frac{1}{(m+1)(m+2)}\,\frac{1}{h}\,(z_r + z_{r-1})(x_r^{m+2} + x_{r-1}^{m+2})$$

$$+ \frac{1}{m+2}\,(z_r x_r^{m+1} - z_{r-1}x_{r-1}^{m+1}). \tag{44}$$

To adapt the first term of the right-hand side, we use the formula*

$$(y+1)^{m+2} = y^{m+2} - E_1\,\frac{(m+2)!}{2!\,m!}\,y^m + E_2\,\frac{(m+2)!}{4!\,(m-2)!}\,y^{m-2} - \ldots$$

$$+ T_1\,\frac{(m+2)!}{1!\,(m+1)!}\,(y+1)^{m+1} - T_2\,\frac{(m+2)!}{3!\,(m-1)!}\,(y+1)^{m-1} + \ldots, \tag{45}$$

* "On the Relations between Bernoulli's and Euler's Numbers," *Quarterly Journal of Mathematics*, 1898, p. 18.

where $E_1, E_2, \ldots, T_1, T_2, \ldots$ are the coefficients in the expansions

$$\left.\begin{aligned} \operatorname{sech}\theta &= 1 - E_1\frac{\theta^2}{2!} + E_2\frac{\theta^4}{4!} - \ldots \\ \tanh\theta &= T_1\frac{\theta}{1!} - T_2\frac{\theta^3}{3!} + \ldots \end{aligned}\right\}. \tag{46}$$

If $$\xi_r = \tfrac{1}{2}(x_{r-1} + x_r),$$

this gives

$$x_r^{m+2} = \xi_r^{m+2} - E_1\frac{(m+2)!}{2!\,m!}(\tfrac{1}{2}h)^2\xi_r^m + E_2\frac{(m+2)!}{4!\,(m-2)!}(\tfrac{1}{2}h)^4\xi_r^{m-2} - \ldots$$

$$+ T_1\frac{(m+2)!}{1!\,(m+1)!}(\tfrac{1}{2}h)\,x_r^{m+1} - T_2\frac{(m+2)!}{3!\,(m-1)!}(\tfrac{1}{2}h)^3 x_r^{m-1} + \ldots,$$

with a similar equation obtained by changing x_r into x_{r-1} and $\frac{1}{2}h$ into $-\frac{1}{2}h$. Multiplying these equations by z_r and z_{r-1} respectively, and adding, substituting in (44), and adding the corresponding expressions for the other trapezia, we find, the summations being from $r = 1$ to $r = p$,

$$\Sigma\int_{x_{r-1}}^{x_r} zx^m\,dx$$

$$= \frac{E_1+1}{1.2}\Sigma\Delta_r\xi_r^m - \frac{E_2-1}{3.4}\,\frac{m!}{2!\,(m-2)!}(\tfrac{1}{2}h)^2\Sigma\Delta_r\xi_r^{m-2} + \ldots$$

$$+\left[\frac{T_2}{2.3}\,\frac{m!}{1!\,(m-1)!}(\tfrac{1}{2}h)^2 zx^{m-1} - \frac{T_3}{4.5}\,\frac{m!}{3!\,(m-3)!}(\tfrac{1}{2}h)^4 zx^{m-3} + \ldots\right]_{x=x_0}^{x=x_p}, \tag{47}$$

so that, when $z_0 = 0$ and $z_p = 0$,

$$P_m = \Sigma\Delta_r\xi_r^m - \tfrac{1}{12}\frac{m!}{2!\,(m-2)!}\Sigma\Delta_r\xi_r^{m-2}h^2 + \tfrac{31}{240}\frac{m!}{4!\,(m-4)!}\Sigma\Delta_r\xi_r^{m-4}h^4 - \ldots, \tag{48}$$

which agrees with (27) in the second term, but not in subsequent terms.

On Aurifeuillians.

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Chapter I., Aurifeuillians.

Chapter II., Bin-Aurifeuillians. Chapter III., Trin-Aurifeuillians.

Notation.—All symbols used in this paper are *integer* numbers: p denotes a *prime*, ω an *odd* number, ϵ an *even* number; λ, μ, ν denote *odd* numbers (except in Art. 17, 18, 31, 32,); A.P.F. denotes *algebraic prime factor*.

Chapter I., Aurifeuillians.

1. *Algebraic Prime Factors* (A.P.F.).

Let n be an *odd* number; n_1, n_2, n_3, &c., *different odd primes*.

Let $\phi(n, U, V)$, or (for shortness), $\phi(n) = (U^n \pm V^n)$,* (1)

so that $\phi(1, U, V)$, or shortly, $\phi(1) = (U \pm V)$.* (1a)

Here $\phi(n)$ may be used, for shortness, whenever the *form* of the functions U, V is not in question.

Then the *algebraic prime factor*, or, for shortness, A.P.F., of the function $\phi(n, U, V)$, or of $\phi(n)$, is defined by

$$\mathfrak{F}\phi(n_1) = \frac{\phi(n_1)}{\phi(1)}, \quad \mathfrak{F}\phi(n_1 n_2) = \frac{\phi(n_1 n_2)\,\phi(1)}{\phi(n_1)\,\phi(n_2)}, \tag{2a}$$

$$\mathfrak{F}\phi(n_1 n_2 n_3) = \frac{\phi(n_1 n_2 n_3)\,\phi(n_1)\,\phi(n_2)\,\phi(n_3)}{\phi(1)\,\phi(n_2 n_3)\,\phi(n_3 n_1)\,\phi(n_1 n_2)}, \text{ &c.,} \tag{2b}$$

and the law of formation is obvious.† When n_1, n_2, n_3, &c., are *not prime*, the function $\mathfrak{F}$ above formed is simply *an* algebraic factor of $\phi(n)$, but not the algebraic *prime* factor thereof.

As above defined, the function $\mathfrak{F}\phi(n, U, V)$ is *independent of the forms* of U, V. When, however, $U = u^\nu$, $V = v^\nu$, and ν is *odd*, a further reduction may be made, and the *Reduced algebraic prime factor* $\mathfrak{F}$ of the function $(u^{n\nu} \pm v^{n\nu})$, *i.e.*, the quantity $\mathfrak{F}\phi(n\nu, u, v)$, may be found

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* There is *no ambiguity* of sign; the sign used in $\phi(n)$ being used *in all its factors* including the A.P.F., *i.e.* in $\mathfrak{F}\phi(n)$.

† In fact $\phi(n, U, V)$ is the Norm of the binomial complex integer $(U - V\rho)$, where ρ is a *proper* n^{th} root of 1.

by a similar procedure. Of course, this quantity is of a *lower order* than the algebraic prime factor $\mathfrak{F}\phi(n, u^\nu, v^\nu)$ previously found. In this way a function $(u^{n\nu} \pm v^{n\nu})$ with *composite* index $(n\nu)$ may be said to have algebraic prime factors of *different orders*, viz.,

$$\mathfrak{F}\phi(n\nu, u, v), \quad \mathfrak{F}\phi(n, u^\nu, v^\nu), \quad \mathfrak{F}\phi(\nu, u^n, v^n),$$

according as it is considered as a function of (u, v), or of (u^ν, v^ν), or of (u^n, v^n).

2. *Aurifeuillians.*—It is proposed to apply the name* *Aurifeuillian* of order $2n$ or n respectively to the following functions (both algebraic prime factors) :—

$$N = \mathfrak{F}(X^{2n} + Y^{2n}), \quad \text{where } 2nXY = z^2 \text{ (a square)}, \tag{3}$$

$$\left.\begin{aligned} N = \mathfrak{F}(X^n + \iota^{n+1} Y^n), \text{ where } nXY = z^2 \text{ (a square)} \\ \text{where } n \text{ is } \textit{odd} \text{ and } \iota = \sqrt{-1}, \text{ so that } \iota^{n+1} = \pm 1 \end{aligned}\right\}. \tag{4}$$

[Here n is supposed to contain *no square* †*factors*; and X, Y are supposed relatively † prime.]

These functions will be styled, for shortness, *Bin-*, *Trin-*, *Quint-*, &c., Aurifeuillians when the *order* = 2, 3, 5, &c.

The condition $2nXY = z^2$, or $nXY = z^2$, taken together with the working condition (X *prime* to Y) involves *one* of the following groups‡ of conditions :—

$X = x^2$, $\quad Y = 2ny^2$; $\quad x$ *odd*, and *prime* to n, y, (3*a*)

$X = n_1 x^2$, $\quad Y = 2n_2 y^2$; $\quad n = n_1 n_2$§; x *odd*; n_1 and x *prime* to n_2 and y, (3*b*)

$X = x^2$, $\quad Y = ny^2$; $\quad x$ prime to n, y, (4*a*)

$X = n_1 x^2$, $\quad Y = n_2 y^2$; $\quad n = n_1 n_2$§; n_1 and x *prime* to n_2 and y. (4*b*)

Thus the most general forms are

$$N = \mathfrak{F}\{(x^2)^{2n} + (2ny^2)^{2n}\}, \tag{5a}$$

$$N = \mathfrak{F}\{(n_1 x^2)^{2n} + (2n_2 y^2)^{2n}\}, \quad \text{where } n_1 n_2 = n, \tag{5b}$$

$$N = \mathfrak{F}\{(x^2)^n + \iota^{n+1}(ny^2)^n\}, \tag{6a}$$

$$N = \mathfrak{F}\{(n_1 x^2)^n + \iota^{n+1}(n_2 y^2)^n\}, \quad \text{where } n_1 n_2 = n, \tag{6b}$$

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* See Art. 3 for explanation of this term.

† These conditions greatly simplify the details, without any real loss of generality.

‡ Hereby N will be an *odd* number, free from unnecessary factors.

§ The working condition as to n above involves n_1, n_2 both free from square factors.

The forms (5*a*), (6*a*) are only simpler forms of the more general forms (5*b*), (6*b*); but, being the more important, are worth separate statement.

From (5*b*), (6*b*) it is seen that, when n is a *composite* odd number, there arise *different species* of Aurifeuillians all of the *same order* ($2n$ or n), the number of species being the same as the number of ways in which $2n$ or n can be resolved into a pair of (mutually prime) factors.

Ex. $\frac{x^{12}+6^6y^{12}}{x^4+6^2y^4}$ and $\frac{3^6x^{12}+2^6y^{12}}{3^2x^4+2^2y^4}$ are Sext-Aurifeuillians;

$\frac{x^{30}+15^{15}y^{30}}{x^{10}+15^5y^{10}}\ \frac{x^2+15y^2}{x^6+15^3y^6}$ and $\frac{3^{15}x^{30}+5^{15}y^{30}}{3^5x^{10}+5^5y^{10}}\ \frac{3x^2+5y^2}{3^3x^6+5^3y^6}$ are Quindec-Aurifeuillians.

In the abbreviated notation of Art. 1, the Aurifeuillians (3), (4) would be written

$$N=\mathfrak{F}(n,\ X^2,\ Y^2),\ \text{or simply}=\mathfrak{F}(n),\ \text{when } 2nXY=\square, \quad (3,\ \text{i})$$

$$\left.\begin{array}{l} N=\mathfrak{f}(n,\ X,\ Y),\quad \text{or simply}=\mathfrak{f}(n),\ \text{when}\quad nXY=\square \\ \text{the sign } (\pm) \text{ between } x^n,\ y^n \text{ being fixed, so that } \epsilon^{N+1}=\pm 1 \end{array}\right\}. \quad (4,\ \text{i})$$

3. *Aurifeuillian Factors* (L, M).—The interest of these numbers lies chiefly in the fact of their being expressible* *algebraically* as the *difference of two squares* (say P^2-Q^2), so that they are *algebraically resolvable* into two† factors ($P\pm Q$). These factors will be styled, for shortness, *Aurifeuillian Factors*, and will be denoted by L, M; so that $N=L.M$: their salient property is

$$\text{If } L=f(x,-y),\ \text{then } M=f(x,+y);\ \text{and } L<M. \qquad (7)$$

4. *Ant-Aurifeuillians.*—Changing Y into $-Y$ in (4), and therewith

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* This property was discovered by M. Aurifeuille, of Toulouse. A table of the values of P and $Q+\sqrt{2nXY}$ or $Q+\sqrt{nXY}$ is given in Ed. Lucas's paper, "Sur la Série récurrente de Fermat," Rome, 1879, p. 6, for the following cases:—

$n = 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33$; $2n = 2, 6, 10, 14, 22, 26, 30, 34$.

From these the factors L, M can be at once found, as $(P\mp Q)$. The values of L, M are also given in recent papers by Mr. C. E. Bickmore, "On the Numerical Factors of (a^n-1)," in the *Messenger of Mathematics*, Vol. XXV., 1896, pp. 1–44, and Vol. XXVI., 1897, pp. 1–37, for the following cases:—

$n = 3, 5, 7, 11, 15$; $2n = 2, 6, 10, 14, 30$.

† By this property, together with the use of "special divisors," it is possible to factorise many very large numbers *completely* (*i.e.*, into their *prime* factors). Many examples will be given in the sequel of numbers of over 20 figures; and some of 30, 40, 50, and (over) 60 figures completely factorised, see Tables, Arts. 16, 18, 31, 32.

y^2 into $-y^2$ in (4*a*, *b*), (6*a*, *b*) a new number* (N) results, which may be styled *Ant-Aurifeuillian*, from its being *algebraically* separable into a *sum of two squares* (say P^2+Q^2), in the same way as the Aurifeuillian itself is separable into a difference of squares, the expressions being in fact conjugate. Here

$$N = \mathfrak{f}\left\{(x^2)^n \quad -\iota^{n+1}(ny^2)^n\right\}, \tag{8a}$$

$$N = \mathfrak{f}\left\{(n_1x^2)^n - \iota^{n+1}(n_2y^2)^n\right\}. \tag{8b}$$

5. *Quadratic Forms.*—In common with the A.P.F. of numbers of form ($X^n \pm Y^n$) in general, all Aurifeuillians can be expressed *algebraically* in various quadratic forms of the type† ($n_1\Xi^2+\iota^{n+1}n_2H^2$), where $n_1n_2 = n$, and generally in more than one way in each type defined by a particular factorization $n_1n_2 = n$ (n_1 prime to n_2).

Moreover the Aurifeuillian Factors L, M can themselves also be expressed *algebraically*‡ in quadratic forms *isomorphic* with each other, and *conformal* with those of N.

Thus, with Aurifeuillians of *odd* order (n),

if $$L = n_1\left\{\phi_1(x,-y)\right\}^2+\iota^{n+1}n_2\left\{\phi_2(x,-y)\right\}^2, \tag{9a}$$

then, by (7), $$M = n_1\left\{\phi_1(x,y)\right\}^2 \quad +\iota^{n+1}n_2\left\{\phi_2(x,y)\right\}^2. \tag{9b}$$

Hence, by *conformal*§ *multiplication*,

$$N = \Phi_1^2+\iota^{n+1}n\Phi_2^2, \quad (\text{since } n = n_1n_2), \tag{10}$$

where $$\Phi_1 = n_1\phi_1(x,-y)\,\phi_1(x,y) \pm n_2\phi_2(x,-y)\,\phi_2(x,y), \tag{10a}$$

$$\Phi_2 = \quad \phi_1(x,-y)\,\phi_2(x,y) \mp \quad \phi_2(x,-y)\,\phi_1(x,y). \tag{10b}$$

(the signs being *unlike*, or *like* in Φ_1, Φ_2 according as ι^{n+1} is + or −).

Results (9*a*, *b*), (10*a*, *b*) apply, with slight modifications, (*e.g.*, omitting ι^{n+1}, &c.,) to Aurifeuillians of *even* order ($2n$).

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* This modification of the Aurifeuillian numbers (4) is due to Mr. Bickmore; see his papers quoted above, *passim*. Note that every Aurifeuillian of *even* order is obviously expressible in form (a^2+b^2) (being itself the A.P.F. of a sum of two squares).

† When $n > 10$, the form $\frac{1}{4}(n_1\Xi^2+\iota^{n+1}n_2H^2)$ has sometimes to be used instead of that given.

‡ This has been shown in the two papers by Mr. Bickmore, quoted above, for most of the values of n used by him; and, from the manner of composition of these numbers from their complex factors, it is seen to be generally true. This property greatly enhances the interest and utility of the Aurifeuillian numbers.

§ *I.e.*, multiplication of two *conformals* $(n_1\phi_1^2+n_2\phi_2^2)\cdot(n_1\psi_1^2+n_2\psi_2^2)$ to form the *conformal* product (10). See the author's paper "On the Connexion of Quadratic Forms," in *Proc. Lond. Math. Soc.*, Vol. XXVIII, 1897, Art. 6–13.

Hence every distinct partition of the factors L, M, defined by a particular set of values of n_1, n_2, ϕ_1, ϕ_2, yields *two* isomorphic partitions of N, in consequence of the double sign ($\pm$) entering into ϕ_1, ϕ_2.

But, further, each distinct pair of values of n_1, n_2 yields in many cases either two or three *alternative* forms of the functions ϕ_1, ϕ_2, and therefore also of the pair L, M, corresponding to the sub-cases when *one* of x, y is even (and the other odd), or *both* of x, y odd; and each such alternative pair of L, M yields—by *conformal multiplication* as above (10, 10a, b)—*two* forms of the product (N) each conformal with L, M. This may be denoted as follows, where similar forms of L, M, N are denoted by like accents,

L', M' yielding N' and $N°$, when y is *even*, (11a)

L'', M'' yielding N'' and $N°$, when x is *even*, (11b)

L''', M''' yielding N''' and $N°$, when x, y are both *odd*; (11c)

or thus

L', M' yielding N' and $N°$, when *one* of x, y is *even*, (11d)

L'', M'' yielding N'' and $N°$, when x, y are *both odd*. (11e)

[The forms differently accented (as L', L'', L''', &c.) are evidently *arithmetically distinct:* they differ, however, only by a unit-factor, the form of which depends on the value of n; *e.g.*, when $n = 3, 5$ the unit-factor is $\frac{1}{4}(1^2+3\,.\,1^2)$, $\frac{1}{4}(3^2-5\,.\,1^2)$ respectively; from this it comes that the products of the similarly accented $L'M'$, $L''M''$, $L'''M'''$ each yield *one* product form with the like accent N', N'', N''', and each yields also the *same* (general) form $N°$.]

Ex.—See Art. **21** for examples of (11 a, b, c), and Art. **30a** for examples of (11 d, e).

6. *Fermat's Divisors.*—This name may be applied to *prime* divisors (p) depending directly on Fermat's theorem. Let

$$\mathfrak{A}_\epsilon = \left\{(n_1x^2)^{2n} + (2n_2y^2)^{2n}\right\}, \quad \text{where } n = n_1n_2, \tag{12}$$

$$\mathfrak{A}_\omega = \left\{(n_1x^2)^{n} + \iota^{n+1}(n_2y^2)^{n}\right\}, \quad \text{where } n = n_1n_2. \tag{13}$$

The prime divisors to be considered are of *two* forms $(2\rho+1)$ and $(4\rho+1)$, where ρ denotes the Aurifeuillian *order*, *i.e.*, $\rho = 2n$ or n in case of $\mathfrak{A}_\epsilon$, $\mathfrak{A}_\omega$ respectively.

6a. *Divisor* $p = (2\rho+1)$. By Fermat's theorem applied to $\mathfrak{A}_\epsilon$, for which $\rho = 2n$,

$$\mathfrak{A}_\epsilon \equiv n_1^{\frac{1}{2}(p-1)} + (2n_2)^{\frac{1}{2}(p-1)}, \pmod p, \text{ [here } x, y \text{ both prime to } p]$$

$$\equiv \left(\frac{n_1}{p}\right) + \left(\frac{2}{p}\right)\left(\frac{n_2}{p}\right), \text{ in Legendre's notation.}$$

Here $n = 2n_1n_2, \quad p = 2\rho+1 = 4n+1.$

Also, if n_1, n_2 be *primes*, then*

$$\left(\frac{n_1}{p}\right) = \left(\frac{n_2}{p}\right) = +1,$$

and, if n_1, n_2 be *composites*, viz.,

$$n_1 = \lambda_1.\lambda_2.\lambda_3\ldots, \quad n_2 = \mu_1.\mu_2.\mu_3\ldots.$$

Then, for every* factor λ_z, μ_z,

$$\left(\frac{\lambda_z}{p}\right) = \left(\frac{\mu_z}{p}\right) = +1,$$

and

$$\left(\frac{2}{p}\right) = -1;$$

therefore

$$\mathfrak{Y}_\epsilon \equiv 0, \pmod p. \qquad (14a)$$

Next, by Fermat's theorem applied to $\mathfrak{Y}_\omega$, for which $\rho = n$,

$$\mathfrak{Y} = \frac{1}{n_1^n}\left\{(n_1x)^{2n} + \iota^{n+1}n^n y^{2n}\right\}$$

$$\equiv \frac{1}{n_1^n}\left\{1 + \iota^{n+1}n^{\frac{1}{2}(p-1)}\right\}, \pmod p, \quad [\text{here } x, y \text{ both prime to } p]$$

$$\equiv \frac{1}{n_1^n}\left\{1 + \iota^{n+1}\left(\frac{n}{p}\right)\right\}, \pmod p.$$

Now suppose $n = ST$, where S contains only factors of form $(4s+1)$, and T contains only factors of form $(4t-1)$, so that

$$S = s_1s_2s_3\ldots s_\alpha, \quad \text{where } s_z = 4\sigma_z+1, \ (z = 1 \text{ to } z = \alpha),$$

$$T = t_1t_2t_3\ldots t_\beta, \quad \text{where } t_z = 4\tau_z-1, \ (z = 1 \text{ to } z = \beta),$$

$$p = 2n+1 = 4\varpi-1, \ (\text{since } n \text{ is } \textit{odd}).$$

Therefore $\left(\frac{s_z}{p}\right) = +1$, for all† values of z; and $\left(\frac{S}{p}\right) = +1$,

$\left(\frac{t_z}{p}\right) = -1$, for all† values of z,

$\left(\frac{T}{p}\right) = +1$; and $n = 4\nu+1$, $\iota^{n+1} = -1$, when β is *even*,

$\left(\frac{T}{p}\right) = -1$; and $n = 4\nu-1$, $\iota^{n+1} = +1$, when β is *odd*.

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* This follows from the law of quadratic reciprocity, noting that n_1, n_2; λ_z, μ_z are factors of $(p-1)$.

† This follows from the law of quadratic reciprocity, noting that s_z, t_z are factors of $(p-1)$.

Hence $\mathfrak{F}_\omega \equiv 0, \pmod p$, in all cases. (14*b*)

Hence, by (14*a*, *b*),

$\mathfrak{F}_\epsilon$ and $\mathfrak{F}_\omega$ are always divisible by $p = 2\rho+1$, (14*c*)

(when x, y prime to p),

and since, by definition, the Aurifeuillian (N) is the A.P.F. of $\mathfrak{F}_\epsilon$ or $\mathfrak{F}_\omega$, it follows* that—

"Either the Aurifeuillian (N), or some one of the other submultiples of $\mathfrak{N}_\epsilon$, $\mathfrak{N}_\omega$, is divisible by $p = 2\rho+1$ (when x, y prime to p)." (14)

6b. *Divisor* $p = (4\rho+1)$. Applying Fermat's theorem to $\mathfrak{F}_\epsilon$, $\mathfrak{F}_\omega$,

$$\mathfrak{F}_\epsilon = \frac{1}{n_1^{2n}}\left\{(n_1x)^{4n} + (2n)^{2n}y^{4n}\right\}, \text{ where } \rho = 2n,\ p = 8n+1$$

$$\equiv \frac{1}{n_1^{2n}}\left\{\left(\frac{n_1}{p}\right)\left(\frac{x}{p}\right) + (2n)^{2n}\left(\frac{y}{p}\right)\right\}, \pmod p,$$

[here x, y both prime to p];

$$\mathfrak{F}_\omega \equiv \frac{1}{n_1^{n}}\left\{(n_1x)^{2n} + \iota^{n+1}.n^n.y^{2n}\right\}, \text{ where } \rho = n,\ p = 4n+1$$

$$\equiv \frac{1}{n_1^{n}}\left\{\left(\frac{n_1}{p}\right)\left(\frac{x}{p}\right) + \iota^{n+1}.n^n\left(\frac{y}{p}\right)\right\}, \pmod p,$$

[here x, y both prime to p].

Now, in both cases, since $p = 4\rho+1 = 8n+1$ in case of $\mathfrak{F}_\epsilon$, or $= 4n+1$ in case of $\mathfrak{F}_\omega$,

$$\left(\frac{n_1}{p}\right) = +1,\dagger \quad \text{and} \quad n^n \equiv +1,\ddagger \pmod p,$$

and $\iota^{n+1} = +1$ or -1, according as $n = 4\nu-1$ or $4\nu+1$; therefore

$$\mathfrak{F}_\epsilon \equiv \frac{1}{(n_1)^{2n}}\left\{\left(\frac{x}{p}\right) + \left(\frac{y}{p}\right)\right\}, \pmod p,$$

$$\mathfrak{F}_\omega \equiv \frac{1}{n_1^{n}}\left\{\left(\frac{x}{p}\right) + \iota^{n+1}\left(\frac{y}{p}\right)\right\}, \pmod p.$$

* Part of this proof was communicated to the author by Mr. Bickmore.

† This follows from the law of quadratic reciprocity.

‡ See *Educational Times*, Vol. LXV., p. 78, Question 13058, by C. E. Bickmore, proving that $(\rho^\rho - 1)$ is divisible by $p = 4\rho+1$.

Hence, using the abbreviations Q.R. = quadratic residue of p, Q.N.R. = quadratic non-residue of p,

$\mathfrak{N}_\epsilon \equiv 0$, (mod p), when one of x, y is Q.R. and the other Q.N.R. of p, (15a)

$\mathfrak{N}_\omega \equiv 0$, (mod p), if $n = 4\nu - 1$, when one of x, y is Q.R., and the other Q.N.R. of p

also, if $n = 4\nu + 1$, when x, y are both Q.R., or both Q.N.R. of p (15b)

Hence it follows that—

"When $\mathfrak{N}_\epsilon$ or $\mathfrak{N}_\omega$ is divisible by $p = (4\rho + 1)$, either the Aurifeuillian (N), or some of the other submultiples of $\mathfrak{N}_\epsilon$, $\mathfrak{N}_\omega$, is divisible by p." (15)

Also, writing N' for the Ant-Aurifeuillian (Art. 4) conjugate to N, and $\mathfrak{N}'$ for the multiple thereof conjugate to N_ω; then, by similar reasoning to above,

$$\mathfrak{N}'_\omega = \left\{(n_1 x^2)^n - \iota^{n+1}(n_2 y^2)^n\right\}, \text{ where } n = n_1 n_2$$

$$\equiv \frac{1}{n^n}\left\{\left(\frac{x}{p}\right) - \iota^{n+1}\left(\frac{y}{p}\right)\right\}, \text{ (mod } p\text{), where } p = 4n+1$$

[here both x, y prime to p]

$\equiv 0$, (mod p), if $n = 4\nu - 1$, when x, y are both Q.R., or both Q.N.R. of p, (15a')

also, if $n = 4\nu + 1$, when one of x, y is Q.R., and the other Q.N.R. of p; (15b')

and therefore

"When $\mathfrak{N}'_\omega$ is divisible by $p = 4\rho + 1$, either the Ant-Aurifeuillian N', or some of the other submultiples of $\mathfrak{N}'_\omega$, is divisible by p." (15')

7. *Quotient-Aurifeuillians.*—If an Aurifeuillian $\mathfrak{A}$ be divisible by another Aurifeuillian (N) of the *same order*, then the quotient (Q) will be, in general, decomposable into two factors, say S, T. For, if $\mathfrak{L}$, $\mathfrak{M}$ and L, M be the Aurifeuillian factors of $\mathfrak{A}$, N respectively, then

$$Q = \mathfrak{A} \div N = \mathfrak{L} \,.\, \mathfrak{M} \div L \,.\, M = S \,.\, T,$$

so that the factors $\mathfrak{L}$, $\mathfrak{M}$ contain between them *all* the factors of L, M.

Also the quadratic forms in which $\mathfrak{A}$, $\mathfrak{L}$, $\mathfrak{M}$; N, L, M are expressible are *conformal* (because $\mathfrak{A}$, N are Aurifeuillians of same

order). Hence also, the quadratic forms in which Q, S, T are expressible are *conformal* with them; and Q, S, T can be actually expressed in those quadratic forms by the process* of *conformal division*, (because they are quotients of conformals). The quotient Q thus has properties quite similar to those of Aurifeuillians.

7a. *Quotient-Aurifeuillians* (continued).—A specially important case of Quotient-Aurifeuillians is obtained by writing

$$X^\omega,\ Y^\omega,\ x^\omega,\ y^\omega \text{ for } X,\ Y,\ x,\ y \text{ respectively,}$$

in the Aurifeuillian functions of Art. 2. Hereby, in the abbreviated notation of (3, i), (4, i),

$$N = \mathfrak{F}(n, X^2, Y^2) \text{ gives } \mathfrak{Y} = \dagger\mathfrak{F}(n, X^{2\omega}, Y^{2\omega}), \qquad (16a)$$

$$N = \mathfrak{F}(n, X, Y) \quad \text{gives } \mathfrak{N} = \dagger\mathfrak{F}(n, X^\omega, Y^\omega). \qquad (16b)$$

Thus the new $\mathfrak{N}$ is seen to be an Aurifeuillian of the *same order* ($2n$ or n) as the old N, and the new $\mathfrak{L}$, $\mathfrak{M}$ are found by changing X, Y, &c., into X^ω, Y^ω, &c. Also $\mathfrak{N}$ is plainly divisible by N (since ω is *odd*). And, in this case, it happens in general that

$$\mathfrak{L} \text{ contains } L \text{ or } M; \quad \text{and } \mathfrak{M} \text{ contains } M \text{ or } L,$$

so that the quotient Q is resolvable into factors (S, T), such that

$$Q = \mathfrak{N} \div N = S.T;\ S = \mathfrak{L} \div L, \text{ or } \mathfrak{L} \div M;\ T = \mathfrak{M} \div M, \text{ or } \mathfrak{M} \div L. \qquad (17)$$

Hence also Q, S, T are expressible (by the process of conformal division) in quadratic forms conformal with those of $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$. And, lastly, the factors S, T are interrelated by the same relation (7), which connects $\mathfrak{L}$ and $\mathfrak{M}$, or L and M. Thus *this* Quotient-Aurifeuillian has all the properties of an Aurifeuillian.

7b. *Reduced Quotient-Aurifeuillians.*—The Quotient-Aurifeuillians (Q), just treated of, are not the *Reduced* A.P.F. of their subject-functions (X^2, Y^2) or (X, Y) with regard to the complete index $n\omega$, except when ω is *prime;* the extension to the case when ω is *composite* is made as follows.

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* *I.e.*, division with *preservation of form;* see Art. 15–23 of the author's paper "On the Connexion of Quadratic Forms," above quoted.

† Here the new $\mathfrak{N}$ is not the A.P.F. of the subject-function with regard to the complete index $n\omega$, but only with regard to the index n, (compare Art. 1).

Let $N(\omega)$* be what the Aurifeuillian functions of Art. 2 become when X^ω, Y^ω, x^ω, y^ω are substituted for X, Y, x, y respectively therein,

$$\left.\begin{aligned} &N(\omega) = \mathfrak{F}(n, X^{2\omega}, Y^{2\omega}) \\ \text{so that}\quad &N(1) = \mathfrak{F}(n, X^2, Y^2) \end{aligned}\right\}, \quad \text{when } 2nXY = \square;$$

$$\left.\begin{aligned} &N(\omega) = \mathfrak{F}(n, X^\omega, Y^\omega) \\ \text{so that}\quad &N(1) = \mathfrak{F}(n, X, Y) \end{aligned}\right\}, \quad \text{when } nXY = \square.$$

Then, changing ω into λ, $\lambda\mu$, $\lambda\mu\nu$, &c. in succession, (λ, μ, ν, &c., being *unequal odd* primes), the rules of Art. 1 give

$$\mathfrak{F}N(\lambda) \quad = N(\lambda) \div N(1), \tag{18a}$$

$$\mathfrak{F}N(\lambda\mu) \quad = N(\lambda\mu)\, N(1) \div N(\lambda)(N\mu), \tag{18b}$$

$$\mathfrak{F}N(\lambda\mu\nu) = \frac{N(\lambda\mu\nu)\, N(\lambda)\, N(\mu)\, N(\nu)}{N(1)\, N(\mu\nu)\, N(\nu\lambda)\, N(\lambda\mu)}, \text{ &c.,} \tag{18c}$$

and the law of formation is obvious. Now each of these functions $\mathfrak{F}$ is (by Art. 1) the *Reduced* A.P.F. of its subject-function with regard to the *complete* index $n\omega$: and is composed of the product of the Quotient-Aurifeuillians described in Art. 7a, divided by a product of similar functions, *e.g.*,

$$\mathfrak{F}N(\lambda\mu\nu) = \left(\frac{N(\lambda\mu\nu)}{N(\mu\nu)}\,\frac{N(\lambda)}{N(1)}\right) \div \left(\frac{N(\nu\lambda)}{N(\nu)}\,\frac{N(\lambda\mu)}{N(\mu)}\right). \tag{18c'}$$

Hence the result, which may be termed a *Reduced Quotient-Aurifeuillian* (Q), is (for reasons similar to those used in Art. 7, 7a) resolvable into two factors (S, T), and the three functions Q, S, T are expressible as before in quadratic forms *conformal* with those of $N(\omega)$ and of its factors $\mathfrak{L}$, $\mathfrak{M}$. And S, T are connected by the relation (7). Thus these Reduced Quotient-Aurifeuillians have the *same* properties as Aurifeuillians.

Examples.—The above properties are of great use in factorising high numbers of form $(u^t \pm v^t)$ where $t = 2\omega$ or ω, and ω is composite. Thus, if $t = 105 = 3 \cdot 5 \cdot 7$, then the *Reduced* A.P.F. of $N = (u^{105} \pm v^{105})$ is only of order

$$(105 + 3 + 5 + 7) - (15 + 21 + 35 + 1) = 48$$

in the subject-functions u, v. And writing $n\lambda\mu = 105$, $N = (u^n + i^{n+1} v^n)$ will be an Aurifeuillian of order n when $nuv = \square$. And the *Reduced* A.P.F. of N, *i.e.*, $\mathfrak{F}N(\lambda\mu)$, will be a Quotient-Aurifeuillian, and therefore resolvable into factors S, T as above. The table shows the resolvable number N.

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* Here $N(\omega)$ is not the A.P.F. of the subject-function with regard to the *complete* index $n\omega$, but only with regard to the index n, (compare Art. 1).

Order $n =$	3,	5,	7,	15,	21,	35,	105
$\lambda\mu =$	35,	21,	15,	7,	5,	3,	1
$N =$	$3^{105}+1$,	$5^{105}-1$,	$7^{105}+1$,	$15^{105}+1$,	$21^{105}-1$,	$35^{105}+1$,	$105^{105}-1$
				$5^{105}+3^{105}$,	$7^{105}-3^{105}$,	$7^{105}+5^{105}$,	$35^{105}-3^{105}$
							$21^{105}-5^{105}$
							$15^{105}-7^{105}$

7c. *Quotient Ant-Aurifeuillians.* — The quotient of one Ant-Aurifeuillian $\mathfrak{A}$ by another (N) of the same order may be treated of in a manner quite similar to that of Art. **7**—**7b**, and the results will be, *mutatis mutandis*, quite similar: the difference being that the quotient (Q) will be expressible as a *sum of two squares* (instead of as a difference of squares = product of two factors). And Q will possess all the properties of an Ant-Aurifeuillian.

Chapter II., Bin-Aurifeuillians.

8. *Bin-Aurifeuillians.*—When, in Results (3 ; 3*a*, *b* ; 5*a*, *b*),

$$n = n_1 = n_2 = 1, \quad X = x^2, \quad Y = 2y^2,$$

the function N becomes the *Bin-Aurifeuillian*

$$N = x^4 + 2^2 . y^4 \text{ [}x \text{ taken}^* \textit{ odd}, \text{ and prime to } y \text{; see (3}a, b\text{)]}, \quad (19)$$

and its quadratic partitions are

$$N^\circ = (x^2)^2 + (2y^2)^2 ; \quad N' = (x^2 - 2y^2)^2 + (2xy)^2. \quad (19a)$$

Its Aurifeuillian factors (L, M), and their quadratic partitions, are

$$L = x^2 - 2xy + 2y^2, \quad M = x^2 + 2xy + 2y^2, \quad (20)$$

$$L' = (x-y)^2 + y^2, \quad M' = (x+y)^2 + y^2. \quad (20a)$$

Also, under Art. **6**, N is divisible by 5 (except when one of x, y contains 5) ; and thus *one* of L, M contains 5, viz.,

$$\left.\begin{array}{l} L \text{ is divisible by 5, when } x+y = 5k, \text{ or } x+2y = 5k \\ M \text{ is divisible by 5, when } x-y = 5k, \text{ or } x-2y = 5k \end{array}\right\}. \quad (21)$$

[It is worth noting that 5 is itself a Bin-Aurifeuillian (since $5 = 1^4 + 4 . 1^4$) ; and its $L = 1$, $M = 5$].

Again, since every 4th power ends in 1, 5, or 6, and x is taken *odd*,

Every Bin-Aurifeuillian ($x^4 + 4y^4$) ends in 1, 5, or 9. (21*a*)

9. *Bin-Aurifeuillian Dimorphs.*—The question now arises whether the Bin-Aurifeuillian partition of a number (N) is *unique*, or whether

* Hereby N will be of form $(4f+1)$.

it may be *dimorphic.* Let, if possible,

$$N = x^4+4y^4 = \xi^4+4\eta^4, \quad (\text{where } \xi \neq x,\ \eta \neq y); \tag{22}$$

therefore $$N = (x^2-2y^2)^2+(2xy)^2 = (\xi^2-2\eta^2)^2+(2\xi\eta)^2. \tag{22a}$$

A *general* solution of these seems difficult: but a somewhat limited solution may be obtained by equating separately the *parts* (a, b) of two partitions (a^2+b^2) of *different* kinds, say

$$\xi^2-2\eta^2 = x^2, \quad \xi\eta = y^2.$$

These involve (ξ, η being supposed *relative primes*)

$$\xi = \tau^2, \quad \eta = v^2, \quad x = t, \quad y = \tau v,$$

whence $$\tau^4-2v^4 = t^2. \tag{23}$$

Any solution of this last (23) gives a solution of (22); now the *general* solution of (23) is known* to be

$$\tau = \tau_0^4+2v_0^4, \quad v = 2\tau_0 v_0 . t_0, \quad \text{where } t_0 = \sqrt{\tau_0^4 \sim 2v_0^4}. \tag{23a}$$

Hence every solution of $\tau_0^4 \sim 2v_0^4 = t_0^2$ gives a solution of (23); and, since t is of the same form as one of the forms of t_0, each set of values of τ, v, t so found from the base-set τ_0, v_0, t_0 may be used as a new base-set (*i.e.*, in place of τ_0, v_0, t_0) from which to generate a *new* set of τ, v, t; and this process may be continued *ad infinitum.* Hence, every base-solution (τ_0, v_0) yields an infinite series of *dimorphic* Bin-Aurifeuillians.

Ex.—The elements of the *lowest* solutions are given in the table below: the two solutions $\tau_0 = 1$, $v_0 = 1$; $\tau_0 = 1$, $v_0 = 13$ are †fundamental, each giving rise to an infinite number of solutions; but the numbers increase in magnitude so very rapidly that it is not worth continuing them.

Data.	Auxiliaries.	Aurifeuillian Partitions.		Bin-Aurifeuillian.
τ_0 ; v_0 ; t_0	τ ; v ; t	x ; y	ξ ; η	N
1 ; 1 ; 1	3 ; 2 ; 7	7 ; 6	9 ; 4	7585
3 ; 2 ; 7	113 ; 84 ; 7967	7967 ; 9492	12769 ; 7056	x^4+4y^4
1 ; 13 ; 239	57123 ; 6214 ; t	t ; τv	τ^2 ; v^2	τ^8+4v^8

* Euler's *Elements of Algebra*, (Hewlett's transl., 1822), Part II, Art. 211.

† One other ($\tau_0 = 1525$, $v_0 = 1343$) is given in Mr. Lebesgue's paper, "Résolution des Équations Biquadratiques," in the *Journal des Math. Pures et Appliquées*, t. XVIII, p. 83.

9a. *Factorisation of above.*—The resolution of these numbers into their Aurifeuillian Factors in *two distinct ways*, $N = L_1M_1 = L_2M_2$, greatly helps their ultimate resolution into *prime* factors; thus, taking the second number above

$$N = 7967^4 + 4\cdot9492^4 = (1525^2 + 9492^2)\,(17459^2 + 9492^2) = L_1 . M_1$$

$$= 12769^4 + 4\cdot7056^4 = (5713^2 + 7056^2)\,(19825^2 + 7056^2) = L_2 . M_2.$$

Here $L_1 = 92{,}423{,}689$, $M_1 = 394{,}914{,}745$; $L_2 = 82{,}425{,}505$; $M_2 = 442{,}817{,}761$. And it will be found that the G.C.M. of L_1, M_2 is 1553, and the G.C.M. of L_2, M_1 is $5 . 277$. Hence it easily results

$$N = (1553 . 59513)\,(5 . 277 . 389 . 733) = (5 . 277 . 59513)\,(389 . 733 . 1553).$$

[This large number, > 36400 billion, would be otherwise difficult to factorise.]

10. *Higher Bin-Aurifeuillians.*—The results of imposing *one** of the conditions

$$\text{i. } x^2 \mp 2y^2 = z^2; \qquad \text{ii. } 2y^2 - x^2 = z^2, \tag{24}$$

on the Bin-Aurifeuillian $N = x^4 + 4y^4 = LM$, are interesting. The results are

$$\left.\begin{aligned} &\text{i. } N = 2x^4 - 2x^2z^2 + z^4 = \mathfrak{L} \text{ of } \mathfrak{N} \\ &\text{ii. } N = 2x^4 + 2x^2z^2 + z^4 = \mathfrak{M} \text{ of } \mathfrak{N} \end{aligned}\right\}, \tag{25}$$

where $\mathfrak{L}$, $\mathfrak{M}$ are the Aurifeuillian Factors of the Higher Bin-Aurifeuillian

$$\mathfrak{N} = z^8 + 4x^8 = \mathfrak{L}\mathfrak{M}.$$

This gives *three* (easily recognizable) cases in which $\mathfrak{N}$ is resolvable into *three* factors, viz.,

$$\text{i. } \mathfrak{N} = LM . \mathfrak{M}; \qquad \text{ii. } \mathfrak{N} = \mathfrak{L} . LM.$$

The general solutions of the Diophantines i, ii of (24), in terms of two arbitrary integers (m, n), take the following four forms†

$$\left.\begin{aligned} &\text{i}a.\ x = m^2 - 2n^2, && y = 2mn, && z = m^2 + 2n^2, && (m \text{ odd}) \\ &\text{i}b.\ x = m^2 + 2n^2, && y = 2mn, && z = m^2 - 2n^2, && (m \text{ odd}) \\ &\text{ii}a.\ x = m^2 + 2mn - n^2, && y = m^2 + n^2, && z = n^2 + 2mn - m^2 \\ &\text{ii}b.\ x = n^2 + 2mn - m^2, && y = m^2 + n^2, && z = m^2 + 2mn - n^2 \end{aligned}\right\}. \tag{26}$$

[m *odd* and n *even* in ii$a$, ii$b$]

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* The results obtained by changing z^2 into $2z^2$ will be found reducible to those given by i and ii.

† Noting that Cases a, b are interconvertible by interchange of x, z.

Examples.—The following are the *lowest* examples of each case:—

Case.	Data.	Parts.	$\mathfrak{N} = (x^4+4y^4)z$.	$\mathfrak{N} = \mathfrak{L}\mathfrak{M}$.	$\mathfrak{L}$ or $\mathfrak{M} = N = LM$.
	m; n	x; y; z		$\mathfrak{L}$; $\mathfrak{M}$	N L M
Ia	1 ; 1	−1 ; 2 ; 3	6.565	65 ; 101	$\mathfrak{L}$ = 5 ; 13
Ib	1 ; 1	3 ; 2 ; −1	26,245	145 ; 181	$\mathfrak{L}$ = 5 ; 29
Ia	1 ; 2	−7 ; 4 ; 9	66,105.925	3.425 ; 19.301	$\mathfrak{L}$ = 25 ; 137
Ib	1 ; 2	9 ; 4 ; −7	177,955.685	7.585 ; 23.461	$\mathfrak{L}$ = 41 ; 185
IIa	1 ; 2	1 ; 5 ; 7	5.764.805	2.305 ; 2.501	$\mathfrak{M}$ = 41 ; 61
IIb	1 ; 2	7 ; 5 ; 1	23.059.205	4.705 ; 4.901	$\mathfrak{M}$ = 29 ; 169
IIa	3 ; 2	17 ; 13 ; 7	27.908.794.565	141.121 ; 197.765	$\mathfrak{M}$ = 185 ; 1069
IIb	3 ; 2	7 ; 13 ; 17	7.008,816,645	60.001 ; 116.645	$\mathfrak{M}$ = 205 ; 569

11. *Bin-Aurifeuillians; Quartic Forms.*—A Bin-Aurifeuillian being a quartic form, it becomes interesting to inquire whether it can be expressed in *other* quartic forms. In the first place, it is known that a Bin-Aurifeuillian (x^4+4y^4) cannot be* a perfect square, and *a fortiori* not a perfect fourth power. The forms considered below are

Form (X^4-Y^4), Art. 11a. Form (X^4+Y^4), Art. 11b,

Form $(U^4+W^4)(V^4+W^4)$, Art. 11c.

11a. *Form* (X^4-Y^4).—Let, if possible,

$$N = x^4+4y^4 = X^4-Y^4. \tag{27}$$

A *general* solution of this seems difficult: but a somewhat limited solution may be obtained by expressing (x^4+4y^4) as a difference of squares, and assuming this expression to be *identical* with the other form X^4-Y^4): thus

$$N = x^2+2y^2{}^2-2xy^2 = X^2{}^2-Y^2{}^2;$$

whence $$x^2+2y^2 = X^2, \quad 2xy = Y^2.$$

These involve (since x, y are supposed *mutually prime*)

$$x = f^2, \quad y = 2g^2, \quad X = h, \quad Y = 2fg,$$

where $$f^4+8g^4 = h^2. \tag{28}$$

Any solution of this last (28) gives a solution of (27): now the *general* solution of (28) may be shown† to be

$$f = f_0^4-8g_0^4, \quad g = 2f_0g_0h_0, \quad \text{where} \quad h_0 = \sqrt{f_0^4+8g_0^4}. \tag{28a}$$

* See Euler's *Elements of Algebra*, Art. 209–1.

† By a process similar to that used in Euler's *Elements of Algebra* (Hewlett's transl.), Part II, Art. 211, for solution of $r^4-2v^4 = t^2$: compare **Art. 9 above.**

Hence every solution of $f_0^4+8g_0^4=h_0^2$ gives a solution of (28); and, since h is of the same form as h_0, each set of values of f, g, h so formed from the base-set f_0, g_0, h_0 may be used as a new base-set (*i.e.*, in place of f_0, g_0, h_0) from which to generate a a *new* set of f, g, h; and this process may be continued *ad infinitum*. Hence every solution f_0, g_0, h_0 yields an infinite series of Bin-Aurifeuillians expressible in the required form X^4-Y^4.

Examples.—The elements of the *lowest* solutions are given in the table below: the solutions $f_0=1$, $g_0=1$; $f_0=239$, $g_0=13$ are fundamental, each giving rise to an infinite number of solutions; but the numbers increase so very rapidly that it is not worth while continuing them.

Data.	Auxiliaries.	Aurifn. Parts.	Quartic Parts.	Bin-Aurifn.
f_0 ; g_0 ; h_0	f ; g ; h	x ; y	X ; Y	N
. ; . ; .	1 ; 1 ; 3	1 ; 2	3 ; 2	65
1 ; 1 ; 3	7 ; 6 ; 113	49 ; 72	113 ; 84	113,260,225
7 ; 6 ; 113	7967 ; 9492 ; $(f^4+8g^4)^{\frac{1}{2}}$	f^2 ; $2g^2$	h ; $2fg$	f^8+64g^8

11a'. *Connexion with Dimorph Aurifeuillians.*—The two Diophantine equations

$$(23),\ t^2=\tau^4\sim 2v^4\ ;\quad (28),\ h^2=f^4+8g^4$$

are known* to be so connected that every solution of either gives one solution of the other. Now the former has been shown to lead to the *dimorph* Bin-Aurifeuillian

$$N_1=x_1^4+4y_1^4=(\tau^2)^4+4(v^2)^4,$$

and the latter has been shown to lead to the Bin-Aurifeuillian

$$N_2=x_2^4+4y_2^4=X^4-Y^4:$$

so that these two kinds of Bin-Aurifeuillians must be connected. Comparing Art. 9, 11, it is seen that

$$\left.\begin{array}{lll} \text{either} & x_1=f,\quad y_1=g\ ; & x_2=f^2=x_1^2\ ;\quad y_2=2g^2=2y_1^2 \\ \text{or} & \tau=h=X,\quad v=Y & \end{array}\right\}. \qquad (29)$$

Thus "Every dimorph Bin-Aurifeuillian (N_1) leads to a Bin-Aurifeuillian (N_2) expressible as (X^4-Y^4), and *vice versa*." (29*a*)

11b. *Form* (X^4+Y^4).—Let, if possible,

$$N=x^4+4y^4=(x^2-2y^2)^2+(2xy)^2=X^4+Y^4.$$

As $Y^2\neq 2y^2$, the algebraic solution involves

$$x^2\sim 2y^2=X^2,\quad 2xy=Y^2.$$

These involve (since x is supposed *odd*, and prime to y)

$$x=t^2,\quad y=2u^2,\quad Y=2tu,\quad t^4\sim 8u^4=X^2.$$

But the latter results, $(t^4\sim 8u^4)$ = square, are known* to be impossible forms: thus a Bin-Aurifeuillian *cannot* take the form (X^4+Y^4).

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* See Mr. Lebesgue's paper quoted above, (footnote † of Art. 9).

11c. *Form* $(U^4+W^4)(V^4+W^4)$.—Let, if possible,

$$N = x^4+4y^4 = (U^4+W^4)(V^4+W^4). \qquad (30)$$

A somewhat limited solution may be obtained* by equating the Aurifeuillian Factors (L, M) of N separately to the factors (U^4+W^4), (V^4+W^4); thus

$$L = (x-y)^2+y^2 = U^4+W^4, \quad M = (x+y)^2+y^2 = V^4+W^4,$$

giving $$x-y = U^2, \quad x+y = V^2, \quad y = W^2,$$

whence $$x = \tfrac{1}{2}(V^2+U^2), \quad y = \tfrac{1}{2}(V^2-U^2) = W^2. \qquad (30a)$$

The *general* solution of the latter Diophantine in terms of two arbitraries (s, t) is

$$U = s^2-2t^2, \quad V = s^2+2t^2, \quad W = 2st, \qquad (30b)$$

whence $$x = s^4+4t^4, \quad y = (2st)^2. \qquad (30c)$$

And, since x is supposed *odd* and prime to y, therefore s must be *odd* and prime to t. Every such pair of values of s, t gives a value of N of the required form; thus the computation of such is *direct*, and the number of such forms is infinite.

Note that x itself is a Bin-Aurifeuillian, and is therefore a multiple of 5 (Art. 8), except when one of s, t contains 5 (in which case y is a multiple of 5): hence N cannot contain 5 in any case.

Examples.—The *lowest* solutions are given in the table below: the numbers will be seen to rise very rapidly; most of the results (N) are so high that it has not been thought worth while to multiply out the product $L \cdot M = N$.

Data.	Quartic Parts.	Aurifn. Parts.	Factors of N.
s ; t	U ; V ; W	x ; y	L ; M
1 ; 1	1 ; 3 ; 2	5 ; 4	17 ; 97
1 ; 2	7 ; 9 ; 4	65 ; 16	2657 ; 6817
1 ; 3	17 ; 19 ; 6	325 ; 36	84,817 ; 131,617
3 ; 1	7 ; 11 ; 6	85 ; 36	3697 ; 15937
3 ; 2	1 ; 17 ; 12	145 ; 144	20,737 ; 104,257
5 ; 1	23 ; 27 ; 10	629 ; 100	289,841 ; 541,441
5 ; 2	17 ; 33 ; 20	689 ; 400	243,521 ; 1345,921

12. *Bin-Aurifeuillian Factors.*—Let N_1, N_2 be two numbers whose product (N) is a Bin-Aurifeuillian. The conditions that N_1, N_2 should be themselves the Aurifeuillian Factors (L, M) of N are

$$N_1 = L = x^2-2xy+2y^2, \quad N_2 = M = x^2+2xy+2y^2. \qquad (31)$$

Hence $$x^2+2y^2 = \tfrac{1}{2}(N_2+N_1), \quad 2xy = \tfrac{1}{2}(N_2-N_1),$$

whence $$x^2-2y^2 = \tfrac{1}{2}\{4N_1N_2-(N_2-N_1)^2\}^{\frac{1}{2}}.$$

* This solution was given by the author in his Paper "On 2 as a 16-ic Residue," Art. 19, in *Proc. Lond. Math. Soc.*, Vol. XXVII.

Hence *three* conditions must obtain that any two numbers N_1, N_2 may be the Aurifeuillian Factors of a Bin-Aurifeuillian (N), viz.,

$$\{4N_1N_2-(N_2-N_1)^2\} = \{2(x^2-2y^2)\}^2 = (2Z)^2, \text{ an } \textit{even square}, \quad (32a)$$

$$(N_1+N_2) \pm 2Z = (2x)^2, \text{ an } \textit{even square}, \quad (32b)$$

$$\tfrac{1}{2}\{(N_1+N_2) \mp 2Z\} = (2y)^2, \text{ an } \textit{even square}. \quad (32c)$$

When N_1, N_2 are given numbers (satisfying the above conditions), the values of x, y giving the Aurifeuillian partition of N, and therewith, also, the quadratic partitions of N, L, M (Art. **5**), are given directly.

13. *Conformal Bin-Aurifeuillian Products.*—The question now arises whether the *product* of two or more Bin-Aurifeuillians can be a Bin-Aurifeuillian; or, conversely, whether a Bin-Aurifeuillian can be resolved into the product of two or more Bin-Aurifeuillians.

Let $$N = x^4+4y^4, \quad N_1 = x_1^4+4y_1^4, \quad N_2 = x_2^4+4y_2^4, \quad (33)$$

and, if possible, let $$N = N_1N_2, \text{ (where } N_1 < N_2). \quad (34)$$

The *general* solution of (34) seems difficult; but a somewhat limited solution may be obtained by first equating the Aurifeuillian Factors (L, M) of N separately to N_1, N_2, thus obtaining *two* equations

$$L = N_1, \quad M = N_2. \quad (35)$$

Let L, M, N_1, N_2 be now expressed in the quadratic form (a^2+b^2); herein the forms of L, M are algebraically *unique* (L', M'), whilst those of N_1, N_2 are algebraically *dimorph* (N_1°, N_1'; N_2°, N_2'); see Art. **5**, **8**. Hence the pair of equations (35) lead to the four systems

$$\textit{Systems}\begin{cases} \text{I. } L' = N_1^\circ, \quad M' = N_2^\circ; & \text{II. } L' = N_1', \quad M' = N_2^\circ \\ \text{III. } L' = N_1^\circ, \quad M' = N_2'; & \text{IV. } L' = N_1', \quad M' = N_2' \end{cases}. \quad (35a)$$

By now equating the *parts* (a, b) of L' separately to the *parts* (a, b) of the corresponding N_1 (*i.e.*, of N_1° or N_1'); and also the *parts* (a, b) of M' separately to the *parts* (a, b) of the corresponding N_2 (*i.e.*, of N_2° or N_2'), each system (I, II, III, IV) yields a group of *four* equations between the six quantities x, y, x_1, y_1, x_2, y_2. Also, in each system, four such groups (of four equations each) may be formed; since *either* part (a, b) of L', M' may be equated to *either* part (a, b) of the corresponding N_1°, N_1', N_2°, N_2'. Also, since the parts (a, b) are defined only by their squares (a^2, b^2), each of the parts (a, b) may have either a $\pm$ sign; so that every group (as above formed) may be

formed in *four* ways, by the mere change of sign of each a, b. Thus finally 64 sets of *four* equations each can be formed, 16 in each system.

The four equations in each of the 64 sets are linear in x, y, and quadratic in x_1, y_1, x_2, y_2. Hence, x, y can always be *linearly* eliminated, thus leaving *two* equations quadratic in x_1, y_1, x_2, y_2. From these two the value of either pair (x_1, y_1), (x_2, y_2) can be solved (by solution of quadratics) in terms of the other pair in a form involving (in the general case) *three* radicals. Thus, *three* conditions must, in general, be satisfied (viz., that the expressions under the three radicals should be perfect squares), in order that this form of solution may be *possible*.

Thus, whenever either of N_1, N_2 is *given*, and also the Aurifeuillian partition (x_1, y_1) or (x_2, y_2) thereof, these criteria can be *directly* applied; and, if satisfied, the other pair (x_2, y_2), or (x_1, y_1), and thence also N_2 or N_1, and, lastly, x, y, and N can be *directly* calculated. So that this problem is *direct*, and the solution is *unique*.

The solution of the problem *ab initio* is, however, much more difficult. In fact, in the 48 cases of Systems II, III, IV, the radicals are such complex functions that a solution seems difficult; but the 16 cases of System I are so simple in form that it is easy to recognize that 12* of the 16 cases involve some impossibility. The remaining *four* cases (here denoted i, ii, iii, iv) of System I turn out to be *completely* solvable, and to be interrelated in an interesting way.

13a. *System I.*—This system is defined (see 35*a*) by

$$L' = N_1^\circ, \quad M' = N_2^\circ,$$

or $(x-y)^2+y^2 = (x_1^2)^2+(2y_1^2)^2$, $(x+y)^2+y^2 = (x_2^2)^2+(2y_2^2)^2$.

The equation of the separate parts ($\pm$a, $\pm$b) gives rise to the four possible Cases i, ii, iii, iv.

i. $x-y = x_1^2$, $x+y = x_2^2$, $y = 2y_1^2 = 2y_2^2$, $x = \frac{1}{2}(x_2^2+x_1^2)$
with *two* conditions $x_2^2-x_1^2 = (2y_1)^2 = (2y_2)^2$ } , (36)

ii. $y-x = x_1^2$, $y+x = x_2^2$, $y = 2y_1^2 = 2y_2^2$, $x = \frac{1}{2}(x_2^2-x_1^2)$
with *two* conditions $x_2^2+x_1^2 = (2y_1)^2 = (2y_2)^2$ } , (36)

iii. $x-y = 2y_1^2$, $x+y = 2y_2^2$, $y = x_1^2 = x_2^2$, $x = y_2^2+y_1^2$
with *two* conditions $y_2^2-y_1 = x_1^2 = x_2^2$ } , (36)

iv. $y-x = 2y_1^2$, $y+x = 2y_2^2$, $y = x_1^2 = x_2^2$, $x = y_2^2-y_1^2$
with *two* conditions $y_2^2+y_1^2 = x_1^2 = x_2^2$ } . (36)

* It seems unnecessary to give the detail.

Thus there are *two* conditions of similar form in each of the four cases,

One of type $y_1 = y_2$ (in i, ii), or $x_1 = x_2$ (in iii, iv);

One of type $x_2^2 \mp x_1^2 = (2y_1)^2$ (in i, ii), or $y_2^2 \mp y_1^2 = x_1^2$ (in iii, iv).

This latter condition is equivalent to $c^2 = a^2 + b^2$.

Now the *general* solution of this Pythagorean equation is expressible in terms of *two* arbitrary integers m, n, viz.,

$$c = m^2 + n^2, \quad a = m^2 \sim n^2 \text{ or } 2mn, \quad b = 2mn \text{ or } m^2 \sim n^2.$$

Here m, n should be so taken that x_1, x_2, x may be all *odd*, and x_1, y_1; x_2, y_2; x, y should be relatively prime; this involves

m, n mutual primes; and either one *odd*, one *even*, or both *odd*. (37*a*)

The general solutions of i, ii, iii, iv are

i. $$\left.\begin{aligned} &x_2 = m^2 + n^2, \quad x_1 = m^2 \sim n^2, \quad y_1 = y_2 = mn \\ &x = m^4 + n^4, \quad y = 2m^2n^2; \quad x_1, x_2, x \text{ all } odd \end{aligned}\right\}, \tag{37}$$

ii. $$\left.\begin{aligned} &2y_1 = 2y_2 = m^2 + n^2, \quad x_1 \text{ or } x_2 = m^2 \sim n^2, \quad x_2 \text{ or } x_1 = 2mn \\ &\text{[This solution rejected, as giving } x_1, x_2, x \text{ all } even\text{]} \end{aligned}\right\}, \tag{37}$$

iii. $$\left.\begin{aligned} &y_2 = m^2 + n^2, \quad x_1 = x_2 = m^2 \sim n^2, \quad y_1 = 2mn \\ &x = m^4 + 6m^2n^2 + n^4, \quad y = (m^2 \sim n^2)^2; \quad x_1, x_2, x \text{ all } odd \end{aligned}\right\}, \tag{37}$$

iv. $$\left.\begin{aligned} &x_1 = x_2 = m^2 + n^2, \quad y_1 \text{ or } y_2 = m^2 \sim n^2 \quad y_2 \text{ or } y_1 = 2mn \\ &x = m^4 - 6m^2n^2 + n^4, \quad y = (m^2 + n^2)^2; \quad x_1, x_2, x \text{ all } odd \end{aligned}\right\}. \tag{37}$$

The solutions when m, n are *both odd* lead eventually only to the *same* results as above, so need not be here detailed; in fact, writing for distinctness m', n' in this case, the two solutions of i, ii, iii, iv are interconvertible by the relations

$$m \text{ or } n = \tfrac{1}{2}(m' + n'), \quad n \text{ or } m = \tfrac{1}{2}(m' \sim n'); \quad m' = m + n, \quad n' = m \sim n.$$

Thus there are only *three* distinct general solutions (Nos. i, iii, iv) practically available. These solutions are *general* and *complete*, as they give the means of *direct* computation of Bin-Aurifeuillians N_1, N_2, N, such that $N_1 . N_2 = N$ for any arbitrary values of m, n (one *odd*, one *even*, and relatively prime).

Examples.—The following table shows the elements and the results of a few of the *lowest* of such numbers $N_1 . N_2 = N$ in each of the solvable cases (i, iii, iv). As N is hereby resolved algebraically into *four* factors, it becomes possible to factorise such numbers (N) *completely*, when $< (9 \text{ million})^4 = 6561 \times 10^{24}$, as the present factor-tables extend to 9 million.

Case.	Data.	Aurifeuillian Parts. Of N_1.	Aurifeuillian Parts. Of N_2.	Aurifeuillians.	Aurif. Parts of N. $[N = N_1 N_2]$
	m ; n	x_1 ; y_1	x_2 ; y_2	N_1 ; N_2	x ; y
i	1 ; 2	3 ; 2	5 ; 2	145 ; 689	17 ; 8
	1 ; 4	15 ; 4	17 ; 4	51,649 ; 84,545	257 ; 32
	1 ; 6	35 ; 6	37 ; 6	1,505,809 ; 1,879,345	1297 ; 72
	3 ; 2	5 ; 6	13 ; 6	5,809 ; 33,745	97 ; 72
	3 ; 4	7 ; 12	25 ; 12	85,345 ; 473,569	337 ; 288
	5 ; 2	21 ; 10	29 ; 10	234,481 ; 747,281	641 ; 200
iii	1 ; 2	3 ; 4	3 ; 5	1105 ; 2581	41 ; 9
	1 ; 4	15 ; 8	15 ; 17	67,009 ; 384,709	353 ; 225
	3 ; 2	5 ; 12	5 ; 13	83,569 ; 114,869	313 ; 25
	3 ; 4	7 ; 24	7 ; 25	1,329,505 ; 1,564,901	1201 ; 49
	5 ; 2	21 ; 20	21 ; 29	834,481 ; 3,023,605	1241 ; 441
iv	1 ; 2	5 ; 4	5 ; 3	1649 ; 949	7 ; 25
	1 ; 4	17 ; 8	17 ; 15	99,905 ; 286,021	161 ; 289
	3 ; 2	13 ; 12	13 ; 5	111,505 ; 31,061	119 ; 169
	3 ; 4	25 ; 24	25 ; 7	1,717,729 ; 400,229	527 ; 625
	5 ; 2	29 ; 20	29 ; 21	1,347,281 ; 1,485,205	41 ; 841

14. *Conformal Product of three Bin-Aurifeuillians.*—Let N_1, N_2, N_3; N_{12}, N_{123} be Bin-Aurifeuillians, defined by

$$\left.\begin{aligned} N_1 &= x_1^4 + 4y_1^4, \quad N_2 = x_2^4 + 4y_2^4, \quad N_3 = x_3^4 + 4y_3^4 \\ N_{12} &= x_{12}^4 + 4y_{12}^4, \quad N_{123} = x_{123}^4 + 4y_{123}^4 \end{aligned}\right\}. \tag{38}$$

Also let N_1, N_2 be such that

$$N_1 . N_2 = N_{12}. \tag{38a}$$

Also, if possible, let N_{12}, N_3 be such that

$$N_{12} . N_3 = N_{123}. \tag{39}$$

Then N_1, N_2, N_{12} are here related precisely as the N_1, N_2, N of Art. **13, 13a**; the N, x, y of that Article being here replaced by N_{12}, x_{12}, y_{12}, the suffixes $_{(12)}$ being introduced to show distinctly that N_{12}, x_{12}, y_{12} arise from N_1, N_2.

Then, multiplying together $(x-y)$, $(x+y)$ of the results (36) of Art. **(13a)**, and changing x, y thereof into the x_{12}, y_{12} of the new notation,

$$\left.\begin{aligned} &\text{Case i.} \quad x_{12}^2 - y_{12}^2 = (x_1 x_2)^2; \qquad \text{Case ii.} \quad y_{12}^2 - x_{12}^2 = (x_1 x_2)^2 \\ &\text{Case iii.} \quad x_{12}^2 - y_{12}^2 = (2y_1 y_2)^2; \qquad \text{Case iv.} \quad y_{12}^2 - x_{12}^2 = (2y_1 y_2)^2 \end{aligned}\right\}. \tag{40}$$

These four results may be included under $x_{12}^2 \sim y_{12}^2 = z^4$ (a square).

Next, taking $\qquad y_3 = x, \quad x_3 = x_{12},$

it will be found that *both* the conditions that $N_{123} = N_{12} \cdot N_3$, *i.e.*, that N_{123}, N_{12}, N_3 should be related similarly to the N, N_1, N_2 of Art. **13a**, are satisfied at once in Cases iii, iv of Art. **13a** (as may be seen by writing x_{12}, y_{12}, x_3, y_3 for x_1, y_1, x_2, y_2 in the conditions there formed); and the x, y of those results (36, iii, iv) become the x_{123}, y_{123} here required. Thus

i. After forming N_1, N_2, N_{12} by Case i, Art. **13a**, then

$$x_{12}^2 - y_{12}^2 = (x_1 x_2)^2; \quad \text{take} \quad y_3 = x_1 x_2, \quad x_3 = x_{12}. \tag{41}$$

Then x_{12}, y_{12}, x_3, y_3 (substituted for x_1, y_1, x_2, y_2 in Case iv, Art. **13a**) satisfy the two conditions of Case iv, Art. **13a**, so that the x, y of the latter become the required

$$x_{123} = y_3^2 - y_{12}^2, \quad y_{123} = x_{12}^2 = x_3^2. \tag{41a}$$

ii. This caseis rejected as leading to x_1, x_2, x_{12}, all *even*; see (37, ii).

iii. After forming N_1, N_2, N_{12} by Case iii, Art. **13a**, then

$$x_{12}^2 - y_{12}^2 = (2y_1 y_2)^2; \quad \text{take} \quad y_3 = 2y_1 y_2, \quad x_3 = x_{12}. \tag{42}$$

Then, x_{12}, y_{12}, x_3, y_3 satisfy the two conditions of Case iv, as before; and similarly

$$x_{123} = y_3^2 - y_{12}^2, \quad y_{123} = x_{12}^2 = x_3^2. \tag{42a}$$

iv. After forming N_1, N_2, N_{12} by Case iv, Art. **13a**, then

$$y_{12}^2 - x_{12}^2 = (2y_1 y_2)^2; \quad \text{take} \quad y_3 = 2y_1 y_2, \quad x_3 = x_{12}. \tag{43}$$

Then x_{12}, y_{12}, x_3, y_3 (substituted for x_1, y_1, x_2, y_2 in Case iii, Art. **13a**) satisfy the two conditions of Case iii, Art. **13a**; and the x, y of the latter become the required

$$x_{123} = y_{12}^2 + y_3^2, \quad y_{123} = x_{12}^2 = x_3^2. \tag{43a}$$

The general values of x_{12}, y_{12}; x_3, y_3; x_{123}, y_{123} in terms of the arbitraries m, n of Art. **13a** are now easily written down; thus the present x_{12}, y_{12} are simply the x, y of Results (37, i, iii, iv), and the rest x_3, y_3, x_{123}, y_{123} can be obtained from above.

Examples.—See end of next Article.

15. *Conformal Continued Product of Bin-Aurifeuillians.*—The law of formation of a Bin-Aurifeuillian ($N_{123\ldots r}$) which shall be the product of any number (r) of Bin-Aurifeuillians ($N_1 N_2 N_3 \ldots N_r$) is now easily seen. Starting with any values of m, n (mutual primes, one odd one

even), and using any of the three known Cases (37, i, iii, iv) to form the first product $N_{12} = N_1 . N_2$; then (by Art. **14**) Cases i, iii lead to Case iv, whilst Case iv leads to Case iii; hence N_3, and N_{123} may be computed as in Art. **14**. The formation of the successive new factors N_4, N_5, &c., and new products N_{1234}, N_{12345}, &c., falls alternately under Cases iii, iv (Art. **14**). It will be seen that the process may be continued *ad infinitum*, and that the result at each step is *unique*, *i.e.*, any pair of values of m, n yields a *definite* continued product.

Examples.—Three examples are given below, starting from the *lowest* values of m, n ($m = 1$, $n = 2$) in each of Cases i, ii, iii, and carried to the fourth order (N_{1234}), showing also the intermediate steps, (thus giving *eight* algebraic factors):

$$N_{12} = N_1 . N_2 ; \quad N_{123} = N_{12} . N_3 = N_1 N_2 N_3 ; \quad N_{1234} = N_{123} . N_4 = N_1 N_2 N_3 N_4. \quad (44)$$

The numbers rise so rapidly that it is not worth while pushing to higher orders. The larger numbers (N_{1234}) here factorised would be otherwise very difficult to resolve completely; (Ex. 3 runs to 26 figures).

Ex. 1.—Take $m = 1$, $n = 2$.

Then, (Art. **13a**, Case i),

$$N_1 = 3^4 + 4 . 2^4 = 145 = 5 . 29 ; \quad N_2 = 5^4 + 4 . 2^4 = 689 = 13 . 53 ;$$

therefore $N_{12} = N_1 N_2 = 17^4 + 4 . 8^4 = 99{,}905 = (5 . 29)(13 . 53)$.

Hence, (Art. **14**, Case iv),

$$N_3 = 17^4 + 4 . 15^4 = 286{,}021 = 229 . 1249 ;$$

therefore $N_{123} = N_{12} . N_3 = 161^4 + 4 . 289^4$.

Hence, (Art. **14**, Case iii),

$$N_4 = 161^4 + 4 . 240^4 = 63{,}841 . 218{,}401$$

$$N_{1234} = N_{123} . N_4 = 141{,}121^4 + 4 . 25{,}921^4$$

$$= (5 . 29)(13 . 53)(229 . 1249)(63{,}841 . 218{,}401).$$

Ex. 2.—Take $m = 1$, $n = 2$.

Then, (Art. **13a**, Case iii),

$$N_1 = 3^4 + 4 . 4^4 = 1105 = (17)(5 . 13) ; \quad N_2 = 3^4 + 4 . 5^4 = 2581 = 29 . 89 ;$$

therefore $N_{12} = N_1 . N_2 = 41^4 + 4 . 9^4 = 2{,}852{,}005 = \{17 . (5 . 13)\} . (29 . 89)$.

Hence, (Art. **14**, Case iv),

$$N_3 = 41^4 + 4 . 40^4 = 13{,}065{,}761 = 1601 . 8161 ;$$

therefore $N_{123} = N_{12} . N_3 = 1519^4 + 4 . 1681^4 = \{17 . (5 . 13)\}(29 . 89)(1601 . 8161)$.

Hence, (Art. **14**, Case iii), $N_4 = 1519^4 + 4 . 134{,}480^4$;*

therefore $N_{1234} = N_{123} . N_4 = 2{,}827{,}361^4 + 4 . 2{,}307{,}361^4$

$$= \{17 . (5 . 13)\}(29 . 89)(1601 . 8161) . N_4.$$

Ex. 3.—Take $m = 1$, $n = 2$.

Then, (Art. **13a**, Case iv),

$$N_1 = 5^4 + 4 . 3^4 = 949 = 13 . 73 ; \quad N_2 = 5^4 + 4 . 4^4 = 1649 = 17 . 97 ;$$

therefore $N_{12} = N_1 . N_2 = 7^4 + 4 . 25^4 = 1{,}564{,}901 = (13 . 73)(17 . 97)$.

* Here $N_4 = L_4 . M_4 = (132{,}961^2 + 134{,}480^2)(135{,}999^2 + 134{,}480^2)$; but the further resolution of L_4, M_4 is not obvious.

Hence, (Art. 14, Case iii),

$$N_3 = 7^4 + 4.24^4 = 1{,}329{,}505 = (5.173).(29.53);$$

therefore $$N_{123} = N_{12}.N_3 = 1201^4 + 4.49^4.$$

Hence, (Art. 14, Case iv),

$$N_4 = 1201^4 + 4.1200^4 = (337.4273).(7{,}204{,}801);$$

therefore $$N_{1234} = N_{123}.N_4 = 1{,}437{,}599^4 + 4.1{,}442{,}401^4$$
$$= (13.73)(17.97)\{(5.173)(29.53)\}.\{(337.4273).(7{,}204{,}801)\}.$$

16. *Quotient Bin-Aurifeuillians.*—Writing 2^ω, x^ω, y^ω for $2, x, y$ in the expression (19); (compare Art. 7),

$$N = x^4 + 2^2y^4 \text{ becomes } \mathfrak{N} = x^{4\omega} + 2^{2\omega}y^{4\omega} = (x^\omega)^4 + 2^2.(2^{\frac{1}{2}(\omega-1)}y^\omega)^4. \quad (45)$$

Hence $\mathfrak{N}$ is a Bin-Aurifeuillian, and it obviously contains N, so that $Q = \mathfrak{N} \div N$ is an algebraic integer, being in fact the A.P.F. of $\mathfrak{N}$ when ω is a *prime*.

One of its (a^2+b^2) quadratic forms is given by *conformal division* (of $\mathfrak{N}$ by N) in the form

$$Q = \frac{\mathfrak{N}}{N} = \left(\frac{x^{2\omega+2} \pm 2^{\omega+1}y^{2\omega+2}}{x^4+2^2y^4}\right)^2 + (2x^2y^2)^2.\left(\frac{x^{2\omega-2} \mp 2^{\omega-1}y^{2\omega-2}}{x^4+2^2y^4}\right)^2, \quad (46)$$

using the upper signs when $\omega = 4\mu+1$, and lower signs when $\omega = 4\mu+3$.

The other (a^2+b^2) quadratic form is also similarly expressible, but the criterion of sign is not so easily expressible in a general form, *i.e.*, except for particular values of ω.

The Aurifeuillian Factors $\mathfrak{L}$, $\mathfrak{M}$ of $\mathfrak{N}$ are

$$\mathfrak{L} = x^{2\omega} - 2^{\frac{1}{2}(\omega+1)}x^\omega y^\omega + 2^\omega y^{2\omega}, \qquad \mathfrak{M} = x^{2\omega} + 2^{\frac{1}{2}(\omega+1)}x^\omega y^\omega + 2^\omega y^{2\omega} \quad (47)$$

$$= (x^\omega - 2^{\frac{1}{2}(\omega-1)}y^\omega)^2 + (2^{\frac{1}{2}(\omega-1)}y^\omega)^2, \quad = (x^\omega + 2^{\frac{1}{2}(\omega-1)}y^\omega)^2 + (2^{\frac{1}{2}(\omega-1)}y^\omega)^2. \quad (48)$$

The factors S, T of Q are the *integer* quotients of $\mathfrak{L}$ by L or M, and of $\mathfrak{M}$ by M or L. To discover which quotients are integral, it suffices to write $x = 1$, $y = 1$ in $\mathfrak{L}$, $\mathfrak{M}$, L, M; whereon L, M reduce to 1, 5; showing* that

$$S = \mathfrak{L} \div L, \quad T = \mathfrak{M} \div M, \quad \text{when } \omega = 8i \pm 1 = \textit{prime},$$
$$\textit{i.e.}, \text{ when } \left(\frac{2}{\omega}\right) = +1, \quad (49a)$$

$$S = \mathfrak{L} \div M, \quad T = \mathfrak{M} \div L, \quad \text{when } \omega = 8i \pm 3 = \textit{prime},$$
$$\textit{i.e.}, \text{ when } \left(\frac{2}{\omega}\right) = -1. \quad (49b)$$

* Adapted from a similar rule for Aurifeuillian Factors, given in Art. 30 of Mr. Bickmore's first paper, above quoted.

Expressing $\mathfrak{X}$, $\mathfrak{M}$, L, M in the (a^2+b^2) quadratic form, S and T can now also be expressed in that form by *conformal division*, in a form similar to (48); but the criterion of sign cannot be so easily expressed in general, *i.e.*, except for particular values of ω.

All the quantities $\mathfrak{Y}$, N; $\mathfrak{X}$, $\mathfrak{M}$, L, M; Q, S, T are also expressible in at least two other quadratic forms depending on the nature of ω, when ω is a *given small* number (Art. **5**).

When ω is not a prime, the Reduced A.P.F. of $\mathfrak{Y}$ may be formed by the formulæ of Art. **7b**, and is a *Reduced Quotient Bin-Aurifeuillian* having the *same* properties as a Bin-Aurifeuillian.

As the quadratic partitions of Q, S, T are completely reducible only when ω is a *given* small number, it is better to pass on at once to the simple cases when $\omega = 3, 5, 7,$ &c., as these alone are of *practical* interest. These may be styled *Cubic*, *Quintic*, &c., *Bin-Aurifeuillians.*

16a. *Cubic Bin-Aurifeuillians.*—Writing $\omega = 3$ in the formulæ of Art. **16**,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{12}+2^6y^{12}}{x^4+2^2y^4} = x^8-2^2x^4y^4+2^4y^8. \tag{50}$$

Here $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$, and therefore also Q, S, T, are capable of the three quadratic partitions (a^2+b^2), (A^2+3B^2), (C^2-3D^2); those of S, T are *monomorph*,* whilst those of Q are* *dimorph*, (since $Q = ST$). The results are

$$Q = (x^4-4y^4)^2+(2x^2y^2)^2 = (x^4-2x^2y^2+4y^4)^2+(2xy)^2(x^2-2y^2)^2 \tag{50a}$$
$$= (x^4-2y^4)^2+3(2y^4)^2 = (x^4-6x^2y^2+4y^4)^2+3(2xy)^2(x^2-2y^2)^2 \tag{50b}$$
$$= (x^4+4y^4)^2-3(2xy)^2 = (x^4+6x^2y^2+4y^4)^2-3(2xy)^2(x^2+2y^2)^2. \tag{50c}$$

$$S = \frac{\mathfrak{L}}{M} = \frac{x^6-4x^3y^3+8y^6}{x^2+2xy+2y^2}; \qquad T = \frac{\mathfrak{M}}{L} = \frac{x^6+4x^3y^3+8y^6}{x^2-2xy+2y^2} \tag{51}$$

$$= (x^4+2x^2y^2+4y^4)-2xy(x^2+2y^2); \qquad = (x^4+2x^2y^2+4y^4)+2xy(x^2+2y^2) \tag{51'}$$
$$= x^2(x-y)^2+y^2(x-2y)^2; \qquad = x^2(x+y)^2+y^2(x+2y)^2 \tag{51a}$$
$$= (x^2-xy-y^2)^2+3y^2(x-y)^2; \qquad = (x^2+xy-y^2)^2+3y^2(x+y)^2 \tag{51b}$$
$$= (x^2-xy+2y^2)^2-3(xy)^2; \qquad = (x^2+xy+2y^2)^2-3(xy)^2. \tag{51c}$$

16b. *Quintic Bin-Aurifeuillians.*—Writing $\omega = 5$ in the formulæ of Art. **16**,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{20}+2^{10}y^{20}}{x^4+2^2y^4} = x^{16}-2^2x^{12}y^4+2^4x^8y^8-2^6x^4y^{12}+2^8y^{16}. \tag{52}$$

Here $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$, and therefore also Q, S, T, are capable of the *three* quadratic partitions (a^2+b^2), (u^2-5v^2), (U^2+5V^2); those of S, T are *monomorph*,* whilst those of Q are *dimorph*,* (since $Q = ST$). The results are

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* Any (C^2-3D^2), (u^2-5v^2) or (W^2-7Z^2) partition can of course be transformed into the *automorphs* $(C_k^2-3D_k^2)$, $(u_k^2-5v_k^2)$, or $(W_k^2-7Z_k^2)$, by conformal multiplication by any integer power (k) of the unit-form $(2^2-3.1^2)$, $(9^2-5.4^2)$, or $(8^2-7.3^2)$ respectively (*see* the author's paper quoted in footnote § to Art. **5**).

$$Q = (x^8 - 2^2x^4y^4 + 2^4y^8)^2 + (2xy)^2(x^4 - 2^2y^4)^2 \quad (52a)$$
$$= (x^8 - 2x^6y^2 - 2^3x^4y^4 - 2^3x^2y^6 + 2^4y^8)^2 + (2xy)^2(x^6 - 2^3y^6)^2 \quad (52b)$$
$$= (x^8 - 2x^4y^4 + 2^4y^8)^2 - 5(2x^4y^4)^2 \quad (52c)$$
$$= (x^8 + 10x^6y^2 + 28x^4y^4 + 40x^2y^6 + 16y^8)^2 - 5(2xy)^2(x^6 + 4x^4y^2 + 8x^2y^4 + 8y^6)^2 \quad (52d)$$
$$= (x^8 - 12x^4y^4 + 16y^8)^2 + 5(2x^2y^2)^2(x^4 - 4y^4)^2 \quad (52e)$$
$$= (x^8 - 10x^6y^2 + 28x^4y^4 - 40x^2y^6 + 16y^8)^2 + 5(2xy)^2(x^6 - 4x^4y^2 + 8x^2y^4 - 8y^6)^2, \quad (52f)$$

$$S = \mathfrak{L} + M = (x^{10} - 2^3x^5y^5 + 2^5y^{10}) + (x^2 + 2xy + 2y^2) \quad (53)$$
$$= x^8 - 2x^7y + 2x^6y^2 - 4x^4y^4 + 8x^2y^6 - 16xy^7 + 16y^8 \quad (53a)$$
$$= (x^4 - x^3y + 2xy^3 - 4y^4)^2 + (xy)^2(x^2 - 2xy + 2y^2)^2 \quad (53b)$$
$$= (x^4 - x^3y + 3x^2y^2 - 2xy^3 + 4y^4)^2 - 5(xy)^2(x^2 - xy + 2y^2)^2 \quad (53c)$$
$$= (x^4 - x^3y - 2x^2y^2 - 2xy^3 + 4y^4)^2 + 5(xy)^2(x^2 - 2y^2)^2, \quad (53d)$$

$$T = \mathfrak{M} + L = (x^{10} + 2^3x^5y^5 + 2^5y^{10}) + (x^2 - 2xy + 2y^2) \quad (53')$$
$$= x^8 + 2x^7y + 2x^6y^2 - 4x^4y^4 + 8x^2y^6 + 16xy^7 + 16y^8 \quad (53a')$$
$$= (x^4 + x^3y - 2xy^3 - 4y^4)^2 + (xy)^2(x^2 + 2xy + 2y^2)^2 \quad (53b')$$
$$= (x^4 + x^3y + 3x^2y^2 + 2xy^3 + 4y^4)^2 - 5(xy)^2(x^2 + xy + 2y^2)^2 \quad (53c')$$
$$= (x^4 + x^3y - 2x^2y^2 + 2xy^3 + 4y^4)^2 + 5(xy)^2(x^2 - 2y^2)^2. \quad (53d')$$

16c. *Septimic Bin-Aurifeuillians.*—Writing $\omega = 7$ in the formulæ of Art. **16**,

$$Q = \frac{\mathfrak{P}}{N} = \frac{x^{28} + 2^{14}y^{28}}{x^4 + 2^2y^4} = x^{24} - 2^2x^{20}y^4 + 2^4x^{16}y^8 - 2^6x^{12}y^{12} + 2^8x^8y^{16} - 2^{10}x^4y^{20} + 2^{12}y^{24}. \quad (54)$$

Here $\mathfrak{P}$, $\mathfrak{L}$, $\mathfrak{M}$, and therefore also Q, S, T, are capable of the *three* quadratic partitions $(a^2 + b^2)$, $(u^2 + 7z^2)$, $(W^2 - 7Z^2)$; those of Q being *dimorph*,* whilst those of S, T are *monomorph** (since $Q = ST$). The results are

$$Q = \{(x^4 - 2^2y^4)(x^8 + 2^4y^8)\}^2 + (2x^2y^2)^2(x^8 - 2^2x^4y^4 + 2^4y^8)^2 \quad (54a)$$
$$= \{x^{12} + 2^6y^{12} - 2x^4y^4(x^4 + 2^2y^4)\}^2 + 7(2x^4y^4)^2(x^4 - 2^2y^4)^2 \quad (54b)$$
$$= (x^4 + 2^2y^4)^6 - 7(x^2y^2)^2(x^8 + 2^2x^4y^4 + 2^4y^8)^2, \quad (54c)$$

$$S = \mathfrak{L} \div L = (x^{14} - 2^4x^7y^7 + 27y^{14}) \div (x^2 - 2xy + 2y^2), \quad (55)$$
$$T = \mathfrak{M} \div M = (x^{14} + 2^4x^7y^7 + 27y^{14}) \div (x^2 + 2xy + 2y^2) \quad (55')$$
$$= x^{12} - 2x^{11}y + 2x^{10}y^2 - 4x^8y^4 + 8x^7y^5 - 8x^6y^6 + 16x^5y^7 - 16x^4y^8 + 32x^2y^{10} - 64xy^{11} + 64y^{12} \quad (55a')$$
$$= x^2(x^5 - x^4y + 2x^2y^3 - 4xy^4 + 4y^5)^2 + y^2(x^5 - 2x^4y + 2x^3y^2 - 4xy^4 + 8y^5)^2 \quad (55b')$$
† $$= (x^6 - x^5y - 3x^4y^2 + 4x^3y^3 - 6x^2y^4 + 4xy^5 + 8y^6)^2 + 7x^2y^2(x^4 - x^3y + 2xy^3 - 4y^4)^2 \quad (55c')$$
† $$= (x^6 - x^5y + 4x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + 8y^6)^2 - 7x^2y^2(x^4 - 2x^3y + 2x^2y^2 - 4xy^3 + 4y^4)^2 \quad (55d')$$

The value of S and the quadratic partitions of S may be obtained by writing $-y$ for y in those of T; the other (dimorph) partitions of Q may be obtained by conformal multiplication of the isomorphs of S, T.

17. *Fermat's Numbers.* (Base 2).—These are numbers of form

$$N_n = 2^n + 1.$$

Writing $x = 1$, $y = 2^\nu$, (where ν may be *any* integer), the Bin-Aurifeuillian $N = x^4 + 4y^4$ becomes

$$N = 1 + 4 \cdot 2^{4\nu} = 1 + 2^{4\nu+2} = N_{4\nu+2}. \quad (56)$$

* *See* the footnote * to Art. **16a**, **16b**.

† These two partitions were communicated by Mr. Bickmore.

Hence *all* Fermat's numbers of this form are resolvable into two factors L, M, of which *one* is always a multiple of 5, (Art. 8),

$$L = (2^{2\nu+1} - 2^{\nu+1} + 1), \quad M = (2^{2\nu+1} + 2^{\nu+1} + 1), \qquad (56a)$$

by the formulæ of Art. 8; and can be completely resolved into their *prime* factors up to $n = 46$ (since at this limit the larger factor $M < 9^*$ million). When $2\nu+1 = \omega = 3, 5, 7,$ &c., they can be completely resolved into their *prime* factors by the formulæ of Arts. **16-16c** up to $n = 66$ (since at this limit the larger factor S or T is still $< 9^*$ million. With the aid of "special divisors"† the resolution into *prime* factors can be carried far beyond this limit.

Examples.—The following Table‡ shows the *complete* factorisation of the *Reduced A.P.F.* of Fermat's Numbers $N_n = 2^n + 1$, where $n = 4\nu + 2$, complete as far as $n = 102$, and—at intervals—as far as $n = 210$. The factorisation of the Aurifeuillian factors L, M or S, T is shown separately, and prime factors § only are shown; the semicolon (;) indicates that the factorisation is *complete.*

[*Note*, that the highest number $(2^{210}+1)$, here *completely* factorised, contains 64 figures.]

FACTORS OF THE REDUCED A.P.F. OF $(2^n + 1)$, [when $n = 4\nu + 2$].

n	L or S	M or T	n	L or S	M or T	n	L or S	M or T
2	1;	5;	26	1613;	53.157;	50	101.8101;	5.268,501;
6	1;	13;	30	61;	1321;	54	246,241;	279,073;
10	5;	41;	34	137.953;	26,317;	58	107,367,629;	536,903,681;
14	113;	29;	38	229.457;	525,313;	62	5581.384,773;	8681.49,477;
18	37;	109;	42	14,449;	1429;	66	312,709;	4,327,489;
22	397;	2113;	46	277.30,269;	1013.1657;	70	47,392,381;	7,416,361;

* The present limit of the large Factor-Tables.

† Reuschle's *Neue zahlentheoretische Tabellen*, Stuttgart, 1856, contains (pp. 42-53) a table of "special prime divisors" of $(2^n - 1)$, up to $p < 5000$. The author has a MS. extension (incomplete) to a much higher limit.

‡ Adapted from the Table at pp. 9, 10 of Lucas's *Memoir* (of 1879) above quoted. The Table is there (p. 7) ascribed to M. Le Lasseur (the results as far as $n = 62$ had been given by M. Landry in his *Décomposition des nombres* $(2^n \pm 1)$ *en leurs facteurs premiers*, Paris, 1869). It has been necessary to *recompute* it entirely, as Lucas does not show the factors of L, M or S, T separately, and shows only the *special* divisors for each value of n, (*i.e.*, divisors not belonging to any *lower* values of n); thus Lucas's Table omits the factor 5 when $n = 10$ and 50, and the factor 13 when $n = 78$. Only one mistake was discovered in the part of Lucas's Table hereby verified, viz., the omission of the factor 12,112,549 when $n = 82$. The large factor of M when $n = 146$ was found in Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 11; it is not in the 1879 Memoir.

§ The determination of most of the large primes (of twelve to sixteen figures) is ascribed to M. Le Lasseur in Ed. Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 11. The author's work has verified the large primes up to 36 millions.

FACTORS OF THE REDUCED A.P.F. OF (2^n+1), [when $n = 4\nu+2$]—*continued.*

n	L or S	M or T
74	149 . 184,481,113 ;	593 . 231,769,777 ;
78	13 . 313 . 1249 ;	3121 . 21,841 ;
82	181,549 . 12,112,549 ;	10169 . 43,249,589 ;
86	1,759,217,765,581 ;	173 . 101,653 . 500,177 ;
90	181 . 54,001 ;	29,249,661 ;
94	140,737,471,578,113 ;	3761 . 7,484,047,069 ;
98	4,981,857,697,937 ;	197 . 19,707,683,773 ;
102	409 . 3061 . 13669 ;	1,326,700,741 ;
110	415,878,438,361 ;	3,630,105,520,141 ;
114	131,101 . 160,969 ;	275,415,303,169 ;
118	1181 . 3541 . 157,649 . 174,877 ;	
122	733 . 1709 . 368,140,581,013 ;	
126	118,750,098,349 ;	40,388,473,189 ;
130	108,140,989,558,681 ;	521 . 51,481 . 34,110,701 ;
138	70,334,392,823,809 ;	5,415,624,023,749 ;
146		293 . 9929 . 649,301,712,182,209 ;
150	63,901 . 13,334,701 ;	1201 . 1,182,468,601 ;
174	349 . 29,581 . 27,920,807,689 ;	
210	1,041,815,865,690,181 ;	421 . 146,919,792,181 ;

The following small Table gives "special divisors" (p) of such of Fermat's Numbers of form $(2^{4\nu+2}+1)$ as are omitted from the previous table, within Lucas's limit $n = 210$. The discovery of a few more such "special divisors" would enable the factorisation of some of these numbers to be completed.

n ...	106 ;	134 ;	142 ;	154 ;	158 ;	162 ;	166 ;	170 ;	178 ;	182 ;
p ...	. ;	269 ;	569 ;	8317 ;	317 ;	. ;	997 ;	1021 . 4421 ;	1069 ;	1093 . 4733 ;

n ...	186 ;	190 ;	194 ;	198 ;	202 ;	206
p ...	373 ;	761 . 54,721 ;	389 . 3881 . 4657 . 5821 ;	. ;	809 ;	. ;

18. *Fermat's Numbers.* (Higher Bases).—Writing in Formulæ (3,3a)

$$X = 1, \quad Y = 2y^3, \quad \text{and} \quad 2n = 2\omega = 2(2\nu+1),$$

ν being any integer, then

$$\mathfrak{A} = 1+Y^{2n} = 1+(2y^2)^{4\nu+2} = 1^4+2^2.\eta^4, \text{ where } \eta = 2^\nu.y^{2\nu+1}. \quad (57)$$

Here $\mathfrak{A}$ is a *Bin-Aurifeuillian*, so that every such number, and its A.P.F., viz.,

$$\mathfrak{A} \div N = (1+Y^{2n}) \div (1+Y^2), \quad \text{where} \quad Y = 2y^3,$$

and also its "Reduced A.P.F." (when further reducible), admit of resolution into the two factors L, M, or S, T, of Arts. **8**, **16**.

The existing Factor-Tables (extending to 9 million) enable the *complete* factorisation (*i.e.*, into *prime* factors) of such numbers to be effected up to the following limits:—

$$\begin{array}{rl} 2n = & 2 \;;\; 6 \;;\; 10 \;;\; 14 \;;\; 18 \;;\; 22 \;;\; 26 \text{ to } 66 \\ Y = 2y^2 = & 2.2121^2 \;;\; 2.38^2 \;;\; 2.5^2 \;;\; 2.2^2 \;;\; 2.2^2 \;;\; 2.2^2 \;;\; 2.1^2 \\ = & 8{,}997{,}282 \;;\; 2888 \;;\; 50 \;;\; 18 \;;\; 8 \;;\; 8 \;;\; 2 \end{array}$$

and, by the aid of "special divisors," *† it can be carried far beyond these limits.

Examples.—The following Table ‡ shows the *complete* factorisation (*i.e.* into *prime* factors) of the following ($Y = 2y^2$ throughout):

$N = 1 + Y^2$ to $Y = 2.40^2 = 3200$,§

$N = (1 + Y^6) + (1 + Y^2)$ to $Y = 2.39^2 = 3042$,

$N = (1 + Y^{2\omega}) + (1 + Y^2)$ when $2\omega = 10, 14, 22, 26$ to various high limits,

$N = (1 + Y^{18}) + (1 + Y^6)$.

The factorisation of the Aurifeuillian factors L, M is shown separately, and *prime*¶ factors only are shown; the semicolon (;) in the Tables of $2\omega = 10, 14, 18, 22, 26$, indicates that the factorisation is *complete*.

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* Reuschle's *Tafeln complexer Primzahlen*, Berlin, 1875, give (p. 467 *et seq.*) the *least* values of Y for which $(1 + Y^{2\omega})$ is divisible by any *given* prime $p < 1000$.

† The author has prepared MS. Tables (which it is hoped to publish hereafter) giving the least value of Y for which $N = (1 + Y^n)$ is divisible by any *given* prime as follows:—

$n = 5$, 10, and 15, $p < 10{,}000$; $n = 7$, $p < 4{,}000$; $n = 11$, $p < 3{,}700$:

by which all prime divisors within these limits can be detected at once for those powers (ω) and for the doubles thereof (*i.e.* 2ω). Most of the large factors (> 1000) of the larger numbers (*i.e.*, when L or $M > 9$ million) here factorised were detected by these MS. tables.

‡ Specially computed by the author for this Paper.

§ The highest number of this kind ($N = 1 + Y^2$) factorisable by the existing Factor-Tables is

$$1 + 8{,}997{,}282^2 = 1^4 + 2^2 . 2121^4 = (8{,}993{,}041)\,(25 . 13 . 27697),$$

and, with the aid of "special divisors," this could be pushed much further. Euler, in *Commentationes Arithmeticæ* (Petropol, 1849), p. 369 *et seq.*, gives a table of the small prime factors of $N = 1 + Y^2$, as far as $Y = 1500$ only: this shows the great advance in factorisation since Euler's time. Note that these numbers ($N = 1 + Y^2$) form a *Bin-Aurifeuillian Chain*, (Art. 20), *i.e.* the M of each number = the L of the next higher number.

¶ The high primes (> 9 million) occurring when $Y = 32, 128, 512$ are given on authority of Lucas's Table (quoted in footnote ‡ to Art. 17). Those occurring among the factors of $(1 + Y^{10})$ were detected, when < 100 million, by the author's MS. Tables above mentioned; and, when > 100 million, by the method developed in Mr. F. W. Lawrence's paper on "Determination of certain Primes" in Vol. XXVIII. of these *Proceedings* (with some improvements communicated to the author in MS. by Mr. Bickmore): this method is *most* effective in factorising and in detecting the prime character of numbers up to about 12 figures, when *given* as factors of a *known* number ($a^n \mp 1$) where a is *small* and n large.

FACTORISATION OF $N = (1 + Y^{2\omega}) \div (1 + Y^2)$, [when $Y = 2y^2$].

y	$Y = 2y^2$	$(1 + Y^2)$		$(1 + Y^6) \div (1 + Y^2)$		$(1 + Y^{10}) \div (1 + Y^2)$	
		L	M	L	M	L	M
1	2	1	5	1	13	5;	41;
2	8	5	13	37	109	41.61;	5.1321;
3	18	13	25	229	457	5.15,101;	145,501;
4	32	25	41	13.61	1321	101.8101;	5.268,501;
5	50	41	61	13.157	3061	5,122,541;	7,622,561;
6	72	61	5.17	13.337	6133	5.4,552,601;	31,720,981;
7	98	5.17	113	8317	13.853	61.1,311,341;	5.701.30,341;
8	128	113	5.29	14,449	13.1429	5.47,392,381;	41.7,416,361;
9	162	5.29	181	23,473	13.37.61	421.1,464,241;	5.41.3,753,661;
10	200	181	13.17	97.373	44,221	1447,960,181;	41.43,120,981;
11	242	13.17	5.53	193.277	64153	5.4561.137341;	881.4,262,941;
12	288	5.53	313	13.5869	37.2437		5.181.281.29,401;
13	338	313	5.73	105,769	123,397	5.	61.881.262,261;
14	392	5.73	421	143,053	13.12697		5.
15	450	421	13.37	189,421	216,481	941.40,768,781;	61.701.1,025,021;
16	512	13.37	5.109	246,241	279,073	5.181.1321.54,001;	41.61.29,247,661;
17	578	5.109	613	13.24229	37.61.157	181.581,433,521;	5.
18	648	613	5.137	13.30553	443,917	5.61.761.718,621;	41.
19	722	5.137	761	13.109.349	549,481	881.15401.19001;	5.821.69,772,181;
20	800	761	29.29	37.16453	13.73.709	41.	
21	882	29.29	25.37	741,721	13.62761	5.	41.61.253,765,901;
22	968	25.37	1013	895,357	13.241.313		5.61.8821.341,521;
23	1058	1013	5.13.17	61.17569	1,169,137		
24	1152	5.13.17	1201	1,272,913	829.1669		
25	1250	1201	1301	13.37.3121	1,626,301		
26	1352	1301	5.281	97.18133	1069.1777		
27	1458	5.281	17.89	109.18793	13.169.693		
28	1568	17.89	125.13	37.97.661	337.7561		
29	1682	125.13	1741	73.37,441	2,928,421		
30	1800	1741	1861	13.193.1249	3,349,861		
31	1922	1861	5.397	13.73.3769	1597.2389		
32	2048	5.397	2113	13.312,709	4,327,489		
33	2178	2113	5.449	1009.4561	13.13.28933		
34	2312	5.449	2381	73.97.733	13.423,457		
35	2450	2381	2521	313.18,637	13.37.12,841		
36	2592	2521	5.13.41	6,534,361	6,907,753		
37	2738	5.13.41	29.97	7,296,769	7,702,069		
38	2888	29.97	5.593	13.241.2593	61.229.613		
39	3042	5.593	3121	37.243,769	73.130,057		
40	3200	3121	3281	9,987,121	13.807,637		

[*Note*, that the highest number $(722^{10} + 1)$, here *completely* factorised, contains 29 figures.]

FACTORISATION OF $N = (1 + Y^{20}) + (1 + Y^2)$, [when $Y = 2y^2$]—*continued.*

y	$Y = 2y^2$	$(1 + Y^{14}) + (1 + Y^2)$		$(1 + Y^{18}) + (1 + Y^6)$	
		L	M	L	M
1	2	113 ;	29 ;	37 ;	109 ;
2	8	29.14449 ;	113.1429 ;	246241 ;	279073
3	18	29.1623823 ;	1373.17837 ;		37.37.25309 ;
4	32	29.47392381 ;	113.7416361 ;	109.181.54001 ;	37.29247661 ;
5	50	29.		37.73.5761801 ;	
6	72	29.1597.3550541;		37.181.	
7	98		29.953	37.73	
8	128	4981857697937 ;	197.19707683773;	37.118750098349;	109.403884731 89;

y	$Y = 2y^2$	$(1 + Y^{22}) + (1 + Y^2)$	
		L	M
1	2	397 ;	2113 :
2	8	2113 . 312,709 ;	397 . 4,327,489 ;
3	18		89 .
4	32	2113 . 415,878,438,361 ;	397 . 3,630,105,520,141 ;
5	50	881 .	

y	$Y = 2y^2$	$(1 + Y^{26}) + (1 + Y^2)$	
		L	M
1	2	1613 ;	53 . 157 ;
2	8	13 . 53 . 157 . 313 . 1249 ;	1613 . 3121 . 21,841 ;
3	18	53 .	
4	32	53 . 157 . 108,140,989,558,681 ;	521 . 1613 . 51,481 . 34,110,701 ;
5	50		53 .

19. *Common Aurifeuillian Factors of Bin-Aurifeuillians.*—It will now be shown that every number F of form (a^2+b^2) is an Aurifeuillian Factor of *four* different Bin-Aurifeuillians, say N_1, N_2, N_3, N_4. It suffices to take

$$\left.\begin{aligned} N_1 &= (b \sim a)^4 + 4a^4, \quad N_2 = (b+a)^4 + 4a^4 \\ N_3 &= (a \sim b)^4 + 4b^4, \quad N_4 = (a+b)^4 + 4b^4 \end{aligned}\right\}, \tag{58}$$

and, resolving N_1, N_2, N_3, N_4 into their Aurifeuillian Factors (L, M),

$$\left.\begin{aligned} N_1 &= \{(b-2a)^2+a^2\}(b^2+a^2), \quad N_2 = (b^2+a^2)\{(b+2a)^2+a^2\} \\ N_3 &= \{(a-2b)^2+b^2\}(a^2+b^2), \quad N_4 = (a^2+b^2)\{(a+2b)^2+b^2\} \end{aligned}\right\}, \tag{59}$$

the common factor (a^2+b^2) is seen to be an Aurifeuillian Factor of *each* of them. Hence many simple relations may be derived between such a set of four Bin-Aurifeuillians, *e.g.*,

$$\left.\begin{aligned} N_1+N_4 &= N_2+N_3 = \quad 6\,(a^2+b^2)^2 = 6F^2 \\ N_2-N_1 &= N_4-N_3 = 8ab\,(a^2+b^2) \;= 8abF \\ N_1-N_3 &= N_2-N_4 = \quad 4\,(a^4-b^4) \end{aligned}\right\}. \qquad (60)$$

In order that the four Bin-Aurifeuillians should fulfil the limitations of Art. 8, *i.e.*, that each x should be *odd* and prime to the corresponding y, it suffices that a, b should be relative primes, one *odd*, one *even*.

The above theorem is true whether F be prime, or composite; but if F be a composite number of form (a^2+b^2), it may be considerably extended. For every factor of F must necessarily be of form $(\alpha^2+\beta^2)$; and F will be expressible in several ways as a sum of squares (a^2+b^2); and the above theorem will be true for *each* of these ways: so that, if r be the number of different ways in which F is expressible as a sum of squares, then F will be a common Aurifeuillian Factor of $4r$ different Bin-Aurifeuillians; and, if $F = a^2+b^2$ be any one of these partitions, the Results (58) give the values of the four Bin-Aurifeuillians corresponding.

Again, since every Bin-Aurifeuillian (N) is the product of two factors (L, M), each of which is of form (a^2+b^2), it follows from the above that every Bin-Aurifeuillian (N) is a common Aurifeuillian Factor of *eight* different Bin-Aurifeuillians. Thus, if

$$N = x^4+4y^4$$

be the base Bin-Aurifeuillian, then, taking its *two-fold* quadratic partition $(a'^2+b'^2)$, $(a''^2+b''^2)$ from $(19a)$, and substituting these for a, b, in the four formulæ (58), the *eight* Bin-Aurifeuillians (say N_1, N_2, N_3, ..., N_8) which contain N as common Aurifeuillian Factor (L or M) are obtained, viz.,

$$\left.\begin{aligned} N_1 &= (2y^2-x^2)^4+4\,(x^2)^4, \quad N_2 = (2y^2+x^2)^4+4\,(x^2)^4 \\ N_3 &= (x^2-2y^2)^4+4\,(2y^2)^4, \quad N_4 = (x^2+2y^2)^4+4\,(2y^2)^4 \\ N_5 &= (2xy-x^2+2y^2)^4+4\,(x^2-2y^2)^4 \\ N_6 &= (2xy+x^2-2y^2)^4+4\,(x^2-2y^2)^4 \\ N_7 &= (x^2-2y^2-2xy)^4+4\,(2xy)^4 \\ N_8 &= (x^2-2y^2+2xy)^4+4\,(2xy)^4 \end{aligned}\right\}, \qquad (61)$$

and writing, as usual, $N_r = L_r M_r$, the *common* Aurifeuillian Factor (N) is $M_1 = L_2 = M_3 = L_4 = M_5 = L_6 = M_7 = L_8 = N$, (61*a*)

and the other Aurifeuillian Factors are, (see Art. 8),

$$\left.\begin{array}{ll} L_1 = x^4+4(y^2\sim x^2)^2, & M_2 = x^4+4(x^2+y^2)^2 \\ L_3 = (x^2\sim 4y^2)^2+4y^4, & M_4 = (x^2+4y^2)^2+4y^4 \\ L_5 = (x^2-2y^2)^2+4(xy-x^2+2y^2)^2, & M_6 = (x^2-2y^2)^2+(xy+x^2-2y^2)^2 \\ L_7 = (x^2-2y^2-4xy)^2+4(xy)^2, & M_8 = (x^2-2y^2+4xy)^2+(2xy)^2 \end{array}\right\}. \quad (61b)$$

Cor.—The above results (61, 61*b*) furnish an easy solution of the problem of Art. 13: for the four Bin-Aurifeuillians N_1, N_2, N_3, N_4 each contain a Bin-Aurifeuillian (N) as *one* of their Aurifeuillian Factors, and the *other* Aurifeuillian Factor (*i.e.*, L_1, M_2, L_3, M_4) of any *one* of them becomes itself a Bin-Aurifeuillian by determining x, y so that

$$\text{one of } (y^2\sim x^2),\ (y^2+x^2),\ (x^2\sim 4y^2),\ (x^2+4y^2) = z^2 \text{ (a square)}; \quad (61c)$$

and these are the conditions of some one or other of the Cases i, ii, iii, iv of Art. 13a, (the x, y, N of this Article corresponding to the x_1, y_1, N_1 or x_2, y_2, N_2 of Art. 13a, with other interchanges of notation, as necessary).

Examples. — The following Table shows the *quartets* of Bin-Aurifeuillians (N_1, N_2, N_3, N_4) which contain the common Aurifeuillian Factors $F = 5$, 13, 17, 25, 37; and also the *octet* of Bin-Aurifeuillians which contain the common Bin-Aurifeuillian Factor $F = 65$.

Factor.	Aurifeuillian Parts.				Bin-Aurifeuillians.			
a ; b ; F	x_1 ; y_1	x_2 ; y_2	x_3 ; y_3	x_4 ; y_4	N_1	N_2	N_3	N_4
1 ; 2 ; 5	1 ; 1	3 ; 1	1 ; 2	3 ; 2	5	85	65	145
3 ; 2 ; 13	1 ; 3	5 ; 3	1 ; 2	5 ; 2	325	949	65	689
1 ; 4 ; 17	3 ; 1	5 ; 1	3 ; 4	5 ; 4	85	629	1105	1649
3 ; 4 ; 25	1 ; 3	7 ; 3	1 ; 4	7 ; 4	325	2725	1025	3425
1 ; 6 ; 37	5 ; 1	7 ; 1	5 ; 6	7 ; 6	629	2405	5809	7585
1 ; 8 ; 65	7 ; 1	9 ; 1	7 ; 8	9 ; 8	2405	6565	18785	22945
7 ; 4 ; 65	3 ; 7	11 ; 7	3 ; 4	11 ; 4	9685	24245	1105	15665

20. *Bin-Aurifeuillian Chains.*—Let a series of Bin-Aurifeuillians N_1, N_2, N_3, ..., N_r be taken, so that

$$N_r = L_r M_r = x_r^4+4y_r^4, \text{ (for each value of } r), \quad (62)$$

and let the parts (x_r, y_r) of the successive functions be taken, so that

Case i.

$$y_1 = y_2 = y_3 = \ldots y_r,$$

$$x_2 = x_1+2y_1,\ x_3 = x_1+4y_1,\ x_4 = x_1+6y_1, \text{ \&c. } \ldots x_r = x_{r-1}+2y_1.$$

Case ii.

$x_1 = x_2 = x_3 = \ldots x_r,$

$y_2 = x_1 + y_1,\ y_3 = 2x_1 + y_1,\ y_4 = 3x_1 + y_1,$ &c. $\ldots\ y_r = x_1 + y_{r-1}.$

Then, in Case i,

$$M_{r-1} = (x_{r-1} + y_{r-1})^2 + y_{r-1}^2 = (x_{r-1} + y_1)^2 + y_1^2,$$
$$L_r = (x_r - y_r)^2 + y_r^2 = (x_{r-1} + y_1)^2 + y_1^2.$$

And, in Case ii,

$$M_{r-1} = (x_{r-1} + y_{r-1})^2 + y_{r-1}^2 = y_r^2 + y_{r-1}^2,$$
$$L_r = (x_r - y_r)^2 + y_r^2 = y_{r-1}^2 + y_r^2.$$

Then, in each Case, $M_{r-1} = L_r,$ (63)

or, in other words, every two *successive* members N_{r-1}, N_r have a *common* Aurifeuillian Factor. Such a series may be called a *Bin-Aurifeuillian Chain*, and has many interesting properties: thus

$$N_r : N_{r-1} = L_r M_r : L_{r-1} M_{r-1} = M_r : L_{r-1}. \quad (64)$$

Similarly, $N_{r-2} : N_{r-3} = M_{r-2} : L_{r-3},$

Hence in every *successive* quartet $(N_{r-3}, N_{r-2}, N_{r-1}, N_r)$

$$N_r N_{r-2} : N_{r-1} N_{r-3} = M_r : L_{r-3}. \quad (64a)$$

Similarly, giving to r every integer value from 1 to $2r$,

$$\frac{N_{2r} N_{2r-2} \ldots N_6 N_4 N_2}{N_{2r-1} N_{2r-3} \ldots N_5 N_3 N_1} = \frac{M_{2r}}{L_1}. \quad (64b)$$

Ex.—See the Table of Factorisation of $(1 + Y^2)$, where $Y = 2y^2$, (Art. 18); these numbers $N = (1^4 + 4y^4)$ will be seen to form a *Bin-Aurifeuillian Chain.*

Chapter III., Trin-Aurifeuillians.

21. *Trin-Aurifeuillians.*—When, in Results (4; 4*a*, *b*; 6*a*, *b*), $n_1 = 1$, $n = n_2 = 3$, $X = x^3$, $Y = 3y^3$, the function N becomes the *Trin-Aurifeuillian*,

$$\left.\begin{array}{l} N = (x^6 + 3^3 y^6) \div (x^2 + 3y^2) = x^4 - 3x^2y^2 + 9y^4 \\ x \text{ being } \textit{prime}^* \text{ to } 3y, \text{ (see Art. 2)} \end{array}\right\}. \quad (65)$$

It has *four* quadratic partitions of form $(A^2 + 3B^2)$, *one* (N°) being

* Hereby N will be of form $(6f + 1)$.

general, and *three* (N', N'', N''') *alternative*, see Art. **5**, viz.,

$$N^\circ = (x^3-3y^3)^2+3\,(xy)^2, \text{ in } all \text{ cases}, \tag{66°}$$

$$N' = (x^3-\tfrac{3}{4}y^3)^2+3\,(\tfrac{3}{4}y^3)^2, \text{ when } y \text{ is } even, \tag{66′}$$

$$N'' = (\tfrac{1}{2}x^3-3y^3)^2+3\,(\tfrac{1}{2}x^3)^2, \text{ when } x \text{ is } even, \tag{66″}$$

$$N''' = \left(\frac{x^3+3y^3}{2}\right)^2+3\left(\frac{x^3-3y^3}{2}\right)^2, \text{ when } x, y \textit{ both odd}. \tag{66‴}$$

Its two Aurifeuillian Factors (L, M) are

$$L = x^3-3xy+3y^2, \quad M = x^3+3xy+3y^2, \tag{67}$$

and are *each* expressible in *three* (alternative) ways

$$L' = (x-\tfrac{3}{2}y)^2+3\,(\tfrac{1}{2}y)^2; \qquad M' = (x+\tfrac{3}{2}y)^2+3\,(\tfrac{1}{2}y)^2; \tag{67′}$$

$$L'' = (\tfrac{1}{2}x)^2+3\,(\tfrac{1}{2}x-y)^2; \qquad M'' = (\tfrac{1}{2}x)^2+3\,(\tfrac{1}{2}x+y)^2; \tag{67″}$$

$$L''' = \left(\frac{x-3y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2; \quad M''' = \left(\frac{x+3y}{2}\right)^2+3\left(\frac{x+y}{2}\right)^2. \tag{67‴}$$

Here L', M', N' apply when y is *even*; L'', M'', N'' when x is *even*;

L''', M''', N''' apply when *both* x, y are *odd*.

Also, by conformal multiplication,

$$L'M' \text{ yields both } N'', N^\circ; \quad L''M'' \text{ yields both } N''', N^\circ;$$

$$L'''M''' \text{ yields both } N''', N^\circ. \tag{68}$$

Thus, when the Aurifeuillian Factors (L, M) of any *particular* Trin-Aurifeuillian (N) are *both prime*, they have *only one* quadratic partition each, of form (A^2+3B^2); and the Trin-Aurifeuillian (N) has in that case only two such partitions (one of which is always N°). If, however, either or both of L, M be *composite*, all their factors must necessarily be of form (A^2+3B^2); and L or M (or both L, M), and therefore also N, will then be polymorphs of form (A^2+3B^2): but these partitions will, as a rule, be only arithmetically (not algebraically) possible: and are therefore not of so much interest as those given. In conformity with the remark at end of Art. **5**, the three analogous forms (N', N'', N'''), or (L', L'', L'''), or (M', M'', M'''), although arithmetically distinct, differ from each other only by the unit-factor $\frac{1}{4}(1^2+3.1^2)$, so that *any two* of a trio can be derived from the third by conformal multiplication by that unit-form: *e.g.*,

$$\tfrac{1}{4}(1^2-3.1^2)\,N' \text{ yields both } N'', N''', \text{ &c., &c.}$$

22. *Trin-Ant-Aurifeuillians.*—When, in Results (4; 4*a*, *b*; 6*a*, *b*; 8*a*, *b*), $n_1 = 1$, $n = n_2 = 3$, $X = x^3$, $Y = -3y^3$, the function N becomes the *Trin-Ant-Aurifeuillian.**

$$\left.\begin{array}{l} N = (x^6 - 3^3.y^6) \div (x^3 - 3y^3) = x^4 + 3x^2y^2 + 9y^4 \\ x \text{ being } \textit{prime}\dagger \text{ to } 3y, \text{ (see Art. 2)} \end{array}\right\}. \qquad (69)$$

It has the following quadratic partitions: one of form (a^2+b^2), one (N°) of form‡ (C^2-3D^2), and three (N', N'', N''') *alternative* of form A^2+3B^2.

$$N = (x^3-3y^3)^2+(3xy)^2, \qquad (70)$$

$$N^\circ = (x^3+3y^3)^2-3(xy)^2, \qquad (70^\circ)$$

$$N' = (x^3+\tfrac{3}{2}y^3)^2+3(\tfrac{3}{2}y^2)^2, \text{ when } y \textit{ is even}, \qquad (70')$$

$$N'' = (\tfrac{1}{2}x^3+3y^3)^2+3(\tfrac{1}{2}x^2)^2, \text{ when } x \textit{ is even}, \qquad (70'')$$

$$N''' = \left(\frac{x^3-3y^3}{2}\right)^2+3\left(\frac{x^2+3y^3}{2}\right)^2, \text{ when } x, y \textit{ both odd}, \qquad (70''')$$

The above forms correspond to the forms (LM), N°, N', N'', N''' of the Trin-Aurifeuillian (Art. **21**) respectively; and the conjugate forms are interconvertible by changing $\pm y^2$ into $\mp y^2$.

23. *Fermat's Divisors.*—Under Art. **6a, b** the conditions of divisibility by 7 and 13 are

Every Trin-Aurifeuillian is divisible by 7 when (and only when) neither of x, y, (x^2+3y^2) contains 7; also $N \equiv 3x^4$, (mod 7), when $x^2+3y^2 \equiv 0$, (mod 7), provided neither of x, y contain 7. (71*a*)

Also $\left.\begin{array}{l} L \text{ contains 7 when } (x+y) \text{ or } (2x-y) = 7k \\ M \text{ contains 7 when } (x-y) \text{ or } (2x+y) = 7k \end{array}\right\}$. (71*b*)

Every Trin-Aurifeuillian is divisible by 13 when xy is a quadratic non-residue of 13, and at the same time neither of x, y, (x^2+3y^2) contains 13. (71*c*)

Every Trin-Ant-Aurifeuillian is divisible by 13 when xy is a quadratic residue of 13, and at the same time neither of x, y, (x^2-3y^2) contains 13. (71*d*)

[It is worth note that both 7 and 13 are Trin-Aurifeuillians; for $7 = (1^6+3^3.1^6) \div (1^2+3.1^2)$, and $13 = (2^6+3^3.1^6) \div (2^2+3.1^2)$; whilst 13 is also a Trin-Ant-Aurifeuillian, for $13 = (1^6-3^3.1^6) \div (1^2-3.1^2)$.]

Again, since the square and the sixth power of every number end in

* This modification of the Trin-Aurifeuillian is due to Mr. Bickmore (see Art. **4**).

† Hereby N will be of form $(12f+1)$.

‡ See footnote * to Art. **16a**.

the *same digit*, which is always one of 0, 1, 4, 5, 6, 9, it follows that

All Trin-Aurifeuillians and all Trin-Ant-Aurifeuillians end in one of the digits 1, 3, 7, 9. (71*e*)

When one of them (N) ends in 3 or 7, the conjugate (N') ends in 7 or 3, so that $N+N' \equiv 0$, (mod 10); also, when one of them ends in 1 or 9, then either $N-N'$ or $N+N' \equiv 0$, (mod 10). (71*f*)

Again, from $66^\circ - 66'''$, and $70^\circ - 70'''$, it follows that—

Every prime factor (F) of a Trin-Aurifeuillian or Trin-Ant-Aurifeuillian is of form $(6f+1)$, $(12f+1)$ respectively. (71*g*)

[*Note*, that properties (71*af*) refer only to the simple functions $N = (x^6 \pm 3^3y^6) + (x^2 \pm 3y^2)$, and not to the Quotient-Functions $(\mathfrak{N}+N)$ treated of in Art. **26**, **26a**, **27**.]

24. *Trin-Aurifeuillian Dimorphs.*—The question now arises whether the Trin-Aurifeuillian partition of a number (N) is *unique*, or whether it may be *dimorphic*.

Let $$N_1 = x_1^4 - 3x_1^2y_1^2 + 9y_1^4, \quad N_2 = x_2^4 - 3x_1^2y_1^2 + 9y_2^4 \tag{72}$$

be, if possible, two *different* Trin-Aurifeuillian forms of the *same* number N; so that $$N = N_1 = N_2. \tag{73}$$

The *general* solution of this seems difficult, but a limited solution may be obtained by identifying two of the *different* quadratic partitions of type (A^2+3B^2) of which N_1, N_2 are capable, Art. **21**; say

$$N_1 = N_1^\circ; \quad N_2 = N_2' \text{ or } N_2'' \text{ or } N_2''' \text{ (alternatively)}. \tag{73a}$$

Thus *three* trial systems arise—

$$\text{i. } N_1^\circ = N_2'; \quad \text{ii. } N_1^\circ = N_2''; \quad \text{iii. } N_1^\circ = N_2'''. \tag{73b}$$

Identifying the *like* quadratic parts (A_1, A_2), (B_1, B_2) separately, each system gives two equations—

$$\text{i. } A_1^\circ = A_2', B_1^\circ = B_2'; \quad \text{ii. } A_1^\circ = A_2'', B_1 = B_2''; \quad \text{iii. } A_1^\circ = B_2''', B_1^\circ = B_2'''. \tag{73c}$$

Referring to Art. **21**, the *first** system gives

$$x_1^2 - 3y_1^2 = x_2^2 - \tfrac{3}{2}y_2^2, \quad x_1y_1 = \tfrac{3}{2}y_2^2, \tag{74}$$

whence $$(x_1 + \tfrac{1}{2}y_1)^2 - 13\left(\frac{y_1}{2}\right)^2 = x_2^2. \tag{74a}$$

These, together with the working conditions, involve

$$y_1, y_2 \text{ both even}; \ x_1, x_2 \text{ both odd, and not multiples of 3.} \tag{74b}$$

* Systems ii, iii appear to involve impossibilities.

The Diophantine (74*a*) has two *general* solutions in terms of two arbitrary quantities m, n, (to be taken relative primes) :—

$$\left.\begin{array}{ll} \text{i}a. & x_1+\tfrac{1}{2}y_1 = m^2+13n^2, \quad \tfrac{1}{2}y_1 = 2mn, \quad x_2 = m^2-13n^2 \\ \text{whence} & x_1 = m^2-2mn+13n^2 = (m-n)^2+3\,(2n)^2 \\ & \qquad\qquad \text{where } m, n \text{ are one } \textit{odd}, \text{ one } \textit{even} \end{array}\right\}. \qquad (75)$$

$$\left.\begin{array}{ll} \text{i}b. & x_1+\tfrac{1}{2}y_1 = \tfrac{1}{2}\,(m^2+13n^2), \quad \tfrac{1}{2}y_1 = mn, \quad x_2 = m^2-13n^2 \\ \text{whence} & x_1 = \tfrac{1}{2}\,(m^2-2mn+13n^2) = \tfrac{1}{2}\left\{(m-n)^2+3\,(2n)^2\right\} \\ & \qquad m, n \text{ both } \textit{odd}\text{; giving } x_1 \text{ even; solution } \textit{rejected} \end{array}\right\}. \qquad (75)$$

Again, the condition $x_1 y_1 = \frac{3}{2}y_2^2$, together with (74*b*), involves

$$x_1 = \xi^2, \quad y_1 = 6\eta^2, \quad y_2 = 2\xi\eta. \qquad (75c)$$

The value of y_1 ($= 4mn$, or $2mn$) shows that either m or n must be a multiple of 3, and, finally, that m, n must have *one* of the forms

$$\text{i}a. \begin{cases} m = 3\mu^2, & \mu^2, & 2\mu^2, & 6\mu^2 \\ n = 2\nu^2, & 6\nu^2, & 3\nu^2, & \nu^2 \end{cases}. \qquad (75b)$$

These give $\qquad \text{i}a. \quad y_1 = 4mn = 6\,(2\mu\nu)^2\text{;} \qquad (75c)$

thus satisfying the condition $y_1 = 6\eta^2$; so that it only remains to determine m, n so as to satisfy the condition $x_1 = \xi^2$, where x_1 is given by (75, i*a*.); but it does not seem easy to do this in a general manner.

*Examples.**—Taking $\mu = 1$, $\nu = 1$; $m = 3\mu^2 = 3$, $n = 2\nu^2 = 2$ in i*a*.

This gives $\quad x_1 = 3^2-2.3.2+13.2^2 = 49$, which satisfies $x_1 = \xi^2$.

Here $\quad x_1 = 49, \; y_1 = 24\text{;} \quad x_2 = 43, \; y_2 = 28.$

Hence $\quad \left.\begin{array}{l} N_1 = L_1 M_1 = 601.7657 = 601\,(13.19.31) \\ N_2 = L_2 M_2 = 589.7813 = (19.31)\,(13.601) \end{array}\right\}$ and $N_1 = N_2$.

25. *Tri - Trin - Ant - Aurifeuillians.* — To investigate whether a number N can be at once a Trin-Aurifeuillian (N_1) and a Trin-Ant-Aurifeuillian N_2. Let, if possible,

$$\left.\begin{array}{l} N = N_1 = N_2, \\ \text{where} \quad N_1 = x_1^2-3x_1^2y_1^2+9y_1^4, \quad N_2 = x_2^2+3x_2^2y_2^2+9y_2^4 \end{array}\right\}. \qquad (76)$$

A general solution seems difficult, but a *limited* solution may be obtained by expressing N_1, N_2 in the various quadratic forms N_1', N_1'', N_1'''; N_2', N_2'', N_2''' of type $N = A^2+3B^2$ of Art. **21**, **22**, and then

* From the general properties of Diophantines, it seems probable that any solution should yield an infinite set of solutions, (compare Art. **9**, **86**); but the author has not been able to determine this case.

equating the parts A_1, B_1 of any one of the forms N_1', N_1'', N_1''' separately to the *corresponding* parts A_2, B_2 of any one of the forms N_2', N_2'', N_2'''. Thus would arise *nine* systems of equations, each of type $A_1 = A_2$, $B_1 = B_2$; or, taking into account the double sign ($\pm$) of A_1, A_2; B_1, B_2, there would be 36 systems of equations, (two in each). Most of these systems appear to be *impossible:* in fact, the only possible systems appear to be those of the *two* groups (I, II) below.

GROUP I. i. $N_1' = N_2'$; ii. $N_1'' = N_2'''$; iii. $N_1''' = N_2''$, (77)

which yield, disregarding the ($\pm$) sign of A, B,

i. $A_1' = A_2'$, $B_1' = B_2'$; ii. $A_1'' = A_2'''$, $B_1'' = B_2'''$; &c. (77*a*)

Taking the values of A, B from Art. **21**, **22**, and introducing the $\pm$ signs, there is found to be *only one* possible case in each system i, ii, iii, and these three possible systems will be found to yield the *same results*, reducible in each case to the *two* conditions

$$x_1^2 - x_2^2 = 3y_1^2 = 3y_2^2. \tag{77b}$$

The further discussion of this case is reserved to Art. **26**.

GROUP II. iv. $N_1' = N_2'''$; v. $N_1'' = N_2''$; vi. $N_1''' = N_2'$, (78)

which yield, disregarding the ($\pm$) sign of A, B,

iv. $A_1' = A_2'''$, $B_1' = B_2''$; &c. &c.

Reducing the A, B, and introducing the $\pm$ sign, as in Group I, there is found *only one* possible case in each system iv, v, vi, and these three possible cases lead to the same results, reducible in each case to the two conditions

$$y_1^2 - y_2^2 = \tfrac{1}{3}x_1^2 = \tfrac{1}{3}x_2^2;$$

but, as this involves x_1, x_2 being each $= 3\xi$, a case expressly excluded (65, 69), this solution is *rejected.*

26. *Various Trin-Aurifeuillians.*—The results of imposing *one* of the conditions

i. $x^2 - 3y^2 = z^2$; ii. $x^2 + 3y^2 = z^2$, (79)

on the Trin-Aurifeuillian N, and Trin-Ant-Aurifeuillian $N`$, where

$$N = x^4 - 3x^2y^2 + 9y^4, \quad N` = {}^*z^4 + 3z^2y^2 + 9y^4, \tag{80}$$

(already partly considered in Art. **25**), are worth further study.

* Note that z is here used (instead of x) in the formula for $N`$: this is done to distinguish it here from the x used in the formula for N.

They give

$$\text{i. } N = N` = x^4 - x^3z^3 + z^4 = \frac{x^6+z^6}{x^2+z^2} = N_{\text{vi}} \text{ (suppose)}, \tag{81}$$

$$\text{ii. } N = z^4 - 3x^2z^2 + 3x^4 = \mathfrak{X} \text{ of } \mathfrak{A}, \quad \text{where } \mathfrak{A} = z^8 - 3z^4x^4 + 9x^8, \tag{82a}$$

$$N` = x^4 - 3x^2z^2 + 3z^4 = \mathfrak{X} \text{ of } \mathfrak{A}, \quad \text{where } \mathfrak{A} = x^8 - 3x^4z^4 + 9z^8. \tag{82b}$$

Thus, Condition ii gives the two higher Trin-Aurifeuillians $\mathfrak{A}$ whose smaller Aurifeuillian Factor $\mathfrak{X}$ is itself of one of the forms N or $N`$.

The results of Condition i are of great interest, viz., that the forms $N, N`, N_{\text{vi}}$ are *interconvertible*,* thus also providing *factorisable* numbers of forms $N`$, and N_{vi},† both being expressible in form $N = LM$.

There are *two* general solutions of each of the Diophantine conditions (i, ii) of (79) in terms of two arbitrary quantities m, n (here $m \neq 3\mu$, and m, n should be relatively prime).

$$\text{i. } x = m^2 + 3n^2, \quad y = 2mn, \quad z = m^2 \sim 3n^2; \quad m, n \text{ one } \textit{odd}, \text{ one } \textit{even}, \tag{83}$$

giving $$L = (m^2 \sim 3mn)^2 + (3mn \sim 3n^2)^2 = a_l^2 + b_l^2,$$

$$M = (m^2 + 3mn)^2 + (3mn + 3n^2)^2 = a_m^2 + b_m^2.$$

Thus $$a_l \sim b_l = a_m \sim b_m = m^2 - 3n^2 = z.$$

$$\text{ia. } x = \tfrac{1}{2}(m^2 + 3n^2), \quad y = mn, \quad z = \tfrac{1}{2}(m^2 \sim 3n^2); \quad m, n \text{ both } \textit{odd}. \tag{83a}$$

These give L, M each $\frac{1}{4}$ of the L, M of Case i.

$$\text{ii. } x = m^2 \sim 3n^2, \quad y = 2mn, \quad z = m^2 + 3n^2; \quad m, n \text{ one } \textit{odd}, \text{ one } \textit{even}. \tag{84}$$

$$\text{iia. } x = \tfrac{1}{2}(m^2 \sim 3n^2), \quad y = mn, \quad z = \tfrac{1}{2}(m^2 + 3n^2); \quad m, n \text{ both } \textit{odd}. \tag{84a}$$

An interesting sub-case under Condition i arises when

$$x^2 - 3y^2 = z^2 \text{ (constant)}. \tag{85}$$

This gives $N = N` = N_{\text{vi}}$, and also gives the *Trin-Aurifeuillian Chains* further developed in Art. **34a**, *q.v.*

Sub-Case (Fermat's Numbers). — The sub-case of the last when $= 1$ is worth separate notice, as giving a factorisable case† of Fermat's Numbers $\mathfrak{A} = (x^6+1)$. Here

$$N = N` = N_{\text{vi}}, \quad \text{where } N_{\text{vi}} = (x^6+1) \div (x^2+1), \tag{86}$$

$$N = x^4 - 3x^2y^2 + 9y^4, \qquad N` = 1 + 3y^2 + 9y^4. \tag{86a}$$

Calling the *successive* numbers $N_1, N_2, N_3, \ldots, N_r$, and applying the subscript r to the *parts* x, y, of N_r, the parts x, y of successive numbers N are connected by the relations

$$x_1^2 - 3y_1^2 = x_2 - 3y_2^2 = \ldots = x_r^2 - 3y_r^2 = 1, \tag{87}$$

* The convertibility of N, $N`$ under Condition i was shown in Art. **25**; the x_2 of that Article is the z of this Article.

† This factorisable form of N_{vi} is due to Mr. Bickmore, and was communicated by him to the Oxford Mathematical Society in February, 1898.

whence $x_r = 2x_{r-1} + 3y_{r-1}$, $y_r = x_{r-1} + 2y_{r-1}$, $(x_1 = 2,\ y_1 = 1)$, (87*a*)

and the m, n of the general formulæ are connected by the relations

$$\left.\begin{aligned} m_{2r} &= n_{2r} + n_{2r-1}, & m_{2r+1} &= n_{2r+1} + 2n_{2r} \\ n_{2r} &= \tfrac{1}{2}(m_{2r-1} + n_{2r-1}), & n_{2r+1} &= m_{2r} + n_{2r} \end{aligned}\right\}, \qquad (87b)$$

These give

$$\left.\begin{aligned} L_r &= (m_r^2 \sim 3m_r n_r)^2 + (3m_r n_r \sim 3n_r^2)^2 = a_{lr}^2 + b_{lr}^2 \\ M_r &= (m_r^2 + 3m_r n_r)^2 + (3m_r n_r + 3n_r^2) = a_{mr}^2 + b_{mr}^2 \end{aligned}\right\}, \qquad (87c)$$

Thus $a_{lr} \sim b_{lr} = a_{mr} \sim b_{mr} = m_r^2 \sim 3n_r^2 = z = 1$, (for all values of r)

so that L_r, M_r each consist of a sum of two *consecutive* squares

These numbers N_r, (being only a sub-case of the last, given by $z = 1$), form a *Trin-Aurifeuillian Chain*, *i.e.*, a series such that $M_{r-1} = L_r$ for all values of r (Art. **34a**).

Examples.—The upper part of the following table shows the factorisation of the numbers $N_{vi} = (x^6+1)+(x^2+1)$, falling under both forms N, N', into their *prime* factors, pushed to a very high limit; the values of L, M show the *Trin-Aurifeuillian Chain* $(M_r = L_{r+1})$.

Below these are given a few more examples, the *lowest* of the numbers $N_{vi} = (x^6+z^6)+(x^2+z^2)$—when $z \neq 1$—also falling under both forms N, N'.

The lower part of the Table contains a few (the *lowest*) examples of Higher Trin-Aurifeuillians $\mathfrak{N}$ whose smaller Trin-Aurifeuillian Factor $\mathfrak{L}$ is itself a Trin-Aurifeuillian, so that $\mathfrak{N} = (L \,.\, M) \,.\, \mathfrak{M}$.

Case.	Data.		Parts.			$N = L \,.\, M$			$N_{vi} = N = N'$
	r	m : n	x :	y :	z :	L	:	M	$(x^6+z^6)+(x^2+z^2)$
First Case, N : $z = 1$.	1	1 : 1	2 :	1 :	1	1	:	13	13
	2	2 : 1	7 :	4 :	1	13	:	181	2,353
	3	5 : 3	26 :	15 :	1	181	:	2521	456,301
	4	7 : 4	97 :	56 :	1	2521	:	13 . 37 . 73	88,519,873
	5	10 : 11	362 :	209 :	1	13 . 37 . 73	:	480,001	13,341,285,133
	6	20 : 15	1351 :	780 :	1	480,001	:	6,811,741	
	7	71 : 41	5042 :	2911 :	1	6,811,741	:	13 . 61 . 181 . 661	
	8	97 : 56	18817 :	10864 :	1	13 . 61 . 181 . 661	:		
Case ii.		1 : 2	13 :	4 :	11	61	:	373	22,753
		3 : 1	14 :	5 :	11	61	:	13 . 37	29,341
		1 : 3	11 :	3 :	13	97	:	349	33,853
		4 : 1	10 :	3 :	13	97	:	1009	97,873

Case iii.	Data.		Parts.			$\mathfrak{N} = \mathfrak{L} \,.\, \mathfrak{M}$			$\mathfrak{N} = \mathfrak{L} \,.\, \mathfrak{M}$ $= L \,.\, M \,.\, \mathfrak{M}$
	m : n		x :	y :	z :	$\mathfrak{L}$	:	$\mathfrak{M}$	
	1 : 1		1 :	1 :	2	1	:	31	217
	2 : 1		1 .	4 .	2	37	:	2551	5,757,607
	1 : 2		11 .	4 .	13	37	:	39901	1,490,475,847
	3 : 1		11 :	4 .	14	37 . 181	:	39901	
	1 : 3		13 :	3 .	14	181	:	5209	

27. *Trin-Aurifeuillian Factors.*—Let N_1, N_2 be two numbers whose product $\mathfrak{N}$ is a Trin-Aurifeuillian. The conditions that N_1, N_2 should be the Aurifeuillian Factors (L, M) of N are

$$N_1 = L = x^3 - 3xy + 3y^3, \quad N_2 = M = x^3 + 3xy + 3y^3. \tag{88}$$

Hence $x^3 + 3y^3 = \frac{1}{2}(N_2 + N_1)$, $3xy = \frac{1}{2}(N_2 - N_1)$,

whence $x^3 - 3y^3 = \frac{1}{6}\{36N_1N_2 - 3(N_2 - N_1)^2\}^{\frac{1}{2}}$.

Hence *three* conditions must obtain that N_1, N_2 may be the Aurifeuillian factors of N, viz.,

$$\{36N_1N_2 - 3(N_2 - N_1)^2\} = \{6(x^3 - 3y^2)^2\} = (6Z)^2, \text{ a } \textit{square}, \tag{88a}$$

$$(N_1 + N_2) \pm 2Z = (2x)^2, \text{ an } \textit{even square}, \tag{88b}$$

$$\tfrac{1}{3}\{(N_1 + N_2) \mp 2Z\} = (2y)^2, \text{ an } \textit{even square}. \tag{88c}$$

When N_1, N_2 are given numbers, (satisfying the above conditions), the values of x, y, the Aurifeuillian parts of N, and therewith also the quadratic partitions $(A^2 + 3B^2)$ of L, M, N, are hereby given directly.

28. *Conformal Trin-Aurifeuillian Products.*—The question now arises whether the *product* of two or more Trin-Aurifeuillians can be a Trin-Aurifeuillian; and, conversely, whether a Trin-Aurifeuillian can be resolved into the product of two or more Trin-Aurifeuillians.

Let

$$N = x^4 - 3x^3y^3 + 9y^4, \quad N_1 = x_1^4 - 3x_1^2y_1^2 + 9y_1^4, \quad N_2 = x_2^4 - 3x_2^2y_2^2 + 9y_2^4, \tag{89}$$

and, if possible, let $$N = N_1 N_2, \text{ (where } N_1 < N_2\text{)}. \tag{90}$$

The *general* solution of (90) seems difficult, but a somewhat limited solution may be obtained by first equating the Aurifeuillian Factors (L, M) of N separately to N_1, N_2, thus obtaining the *two* equations

$$L = N_1, \quad M = N_2. \tag{90a}$$

Let L, M; N_1, N_2 be now expressed in the quadratic form $(A^2 + 3B^2)$; herein the forms of L, M are each *trimorph*, whilst N_1, N_2 are each *tetramorph*, (Art. 21). Observing that a corresponding pair (L', M'), (L'', M''), (L''', M''') must necessarily enter together into (90), and, therefore, into any one form of Equation (90*a*), whilst the forms of N_1, N_2 are *independent*, the pair of equations (90*a*) may be expressed in 48 ways, which are summarized below.

$L' = \textit{any}$ of $N_1^\circ, N_1', N_1'', N_1'''$; $M' = \textit{any}$ of $N_2^\circ, N_2', N_2'', N_2'''$, (16 pairs).

$L'' = \textit{any}$ of $N_1^\circ, N_1', N_1'', N_1'''$; $M'' = \textit{any}$ of $N_2^\circ, N_2', N_2'', N_2'''$, (16 pairs).

$L''' = \textit{any}$ of $N^\circ, N_1', N_1'', N_1'''$; $M''' = \textit{any}$ of $N_2^\circ, N_2', N_2'', N_2'''$, (16 pairs).

By now equating the parts (A, B) of any corresponding pair (L', M'), (L'', M''), (L''', M''') separately to the parts (A, B) of each of the four forms of N_1, N_2 respectively, each of the 48 systems above yields *one* group of *four* equations between the six quantities x, y, x_1, y_1, x_2, y_2. Further, as each of the parts A, B may have either a $\pm$ sign, each of the above 48 groups may be formed in *four* ways.

The four equations in each of the 192 sets are *linear* in x, y, and *quadratic* in x_1, y_1, x_2, y_2. Hence x, y can always be *linearly* eliminated, thus leaving *two* equations *quadratic* in x_1, y_1, x_2, y_2. From these the value of either pair (x_1, y_1), (x_2, y_2) can be solved (by quadratics) in terms of the other pair in a form involving (in the general case) *three* radicals. Thus *three* conditions must in general be satisfied (viz., that the expressions under the three radicals should be perfect squares) in order that this form of solution may be *possible*.

Thus, whenever either N_1 or N_2 is *given*, and also the Aurifeuillian parts (x_1, y_1) or (x_2, y_2) thereof, these criteria can be *directly* applied; and, if satisfied, the other pair (x_2, y_2) or (x_1, y_1) and thence also N_2 or N_1, and, lastly, x, y, and N, can be *directly* calculated, so that this problem is *direct*, and the solution is *unique*.

The solution of the problem *ab initio* is, however, much more difficult.

In a good many of the 192 sets of equations above specified it is found, on attempting the solution, that some impossibility* is involved. The following are the only cases which have been found completely solvable:—

$$\left.\begin{array}{lll}
\text{i.} & L' = N_1', \quad M' = N_2', & (y, y_1, y_2 \text{ all even}) \\
\text{ii.} & L' = N_1'', \quad M' = N_2'', & (y, x_1, x_2 \text{ all even}) \\
\text{iii.} & L''' = N_1', \quad M''' = N_2''', & (y_1 \text{ even, the rest odd}) \\
\text{iv.} & L''' = N_1''', \quad M''' = N_2', & (y_2 \text{ even, the rest odd}) \\
\text{v.} & L''' = N_1'', \quad M''' = N_2''', & (x_1 \text{ even, the rest odd}) \\
\text{vi.} & L''' = N_1''', \quad M''' = N_2'', & (x_2 \text{ even, the rest odd})
\end{array}\right\} . \quad (91)$$

28a. The detailed solutions are—

***System* i.** $(y, y_1, y_2$ all even$)$.—The equations are

$$(x-\tfrac{3}{4}y)^3+3\left(\frac{y}{2}\right)^2 = L' = N_1' = (x_1^2-\tfrac{3}{2}y_1^2)^3+3(\tfrac{3}{2}y_1^2)^2,$$

$$(x+\tfrac{3}{4}y)^3+3\left(\frac{y}{2}\right)^2 = M' = N_2' = (x_2^2-\tfrac{3}{2}y_2^2)+3(\tfrac{3}{2}y_2^2)^2.$$

* It seems unnecessary to give the detail.

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$x-\tfrac{3}{2}y = x_1^2-\tfrac{3}{2}y_1^2, \quad x+\tfrac{3}{2}y = x_2^2-\tfrac{3}{2}y_2^2, \quad y = 3y_1^2 = 3y_2^2,$$

giving $\quad x = \tfrac{1}{3}(2x_1^2+x_2^2)$

with the *two* conditions $x_2^2-x_1^2 = (3y_1)^2 = (3y_2)^2$ } . (92, i)

By varying the signs of the parts $(\pm A)$, $(\pm B)$ equated, another possible case is found ; but the results differ from the above only in the interchange of x_1, x_2, and change of sign of x.

System ii. (y, x_1, x_2 all even).—The equations are

$$(x-\tfrac{3}{2}y)^2+3\left(\frac{y}{2}\right)^2 = L' = N_1'' = \left(\frac{x_1^2}{2}-3y_1^2\right)^2+3\left(\frac{x_1^2}{2}\right)^2,$$

$$(x+\tfrac{3}{2}y)^2+3\left(\frac{y}{2}\right)^2 = M' = N_2'' = \left(\frac{x_2^2}{2}-3y_2^2\right)^2+3\left(\frac{x_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A)$, $(\pm B)$ as follows :—

$$x-\tfrac{3}{2}y = \tfrac{1}{2}x_1^2-3y_1^2, \quad x+\tfrac{3}{2}y = \tfrac{1}{2}x_2^2-3y_2^2, \quad y = x_1^2 = x_2^2,$$

giving $\quad x = -(y_1^2+2y_2^2)$

with the *two* conditions $y_1^2-y_2^2 = x_1^2 = x_2^2$ } . (92, ii)

By varying the signs of the parts $(\pm A)$, $(\pm B)$ equated, another possible case is found ; but the results differ from the above only in the interchange of y_1, y_2, and change of sign of x.

System iii. (y_1 even, the rest odd).—The equations are

$$\left(\frac{x-3y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2 = L''' = N_1' = (x_1^2-\tfrac{3}{2}y_1^2)^2+3(\tfrac{3}{2}y_1^2)^2,$$

$$\left(\frac{x+3y}{2}\right)^2+3\left(\frac{x+y}{2}\right)^2 = M''' = N_2''' = \left(\frac{x_2^2+3y_2^2}{2}\right)^2+3\left(\frac{x_2^2-3y_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$-(x-3y) = 2x_1^2-3y_1^2, \qquad x-y = 3y_1^2,$$

$$x+3y = x_2^2+3y_2^2, \qquad -(x+y) = x_2^2-3y_2^2,$$

giving $\quad x = y_2^2+2y_1^2, \quad y = x_1^2 = x_2^2$

with *two* conditions $\quad y_2^2-y_1^2 = x_1^2 = x_2^2$ } . (92, iii)

System iv.—The results differ from System iii only in the interchange of (x_1, y_1) with (x_2, y_2), and change of sign of x.

System v. (x_1 even, the rest odd).—The equations are

$$\left(\frac{x-3y}{2}\right)^3+3\left(\frac{x-y}{2}\right)^2 = L''' = N_1''' = (\tfrac{1}{2}x_1^2-3y_1^2)+3(\tfrac{1}{2}x_1^2)^3,$$

$$\left(\frac{x+3y}{2}\right)^3+3\left(\frac{x+y}{2}\right)^2 = M''' = N_2''' = \left(\frac{x_2^2+3y_2^2}{2}\right)^2+3\left(\frac{x_2^2-3y_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$x-3y = x_1^2-6y_1^2, \quad x-y = x_1^2,$$

$$x+3y = x_2^2+3y_2^2, \quad x+y = x_2^2-3y_2^2,$$

giving $x = \frac{1}{3}(2x_1^2+x_2^2), \quad y = 3y_1^2 = 3y_2^2$

with *two* conditions $x_2^2-x_1^2 = (3y_1)^2 = (3y_2)^2$ } . (92, v)

System vi.—The results differ from System v only in the interchange of (x_1, y_1) with (x_2, y_2), and change of sign of y.

The *two* conditions in all these systems are very similar, viz.,

One of type $y_1 = y_2$ (in i, v, vi) ; or $x_1 = x_2$ (in ii, iii, iv).

One of type $x_2^2 \sim x_1^2 = (3y_1)^2$, (in i, v, vi) ; or $y_2^2 \sim y_1^2 = x_1^2$ (in ii, iii, iv).

The latter condition is equivalent to $c^2 = a^2+b^2$. There are *two* general solutions of this Pythagorean equation in terms of two arbitrary quantities (m, n) (*cf.* Art. **13a**); but *one* of the general solutions in each system is found (on actual trial) to violate some of the initial (arithmetical)* conditions as to *oddness* or *evenness* of some of the quantities x_1, y_1, x_2, y_2, leaving effectively *only one* general solution of each system; and also involving effectively

m, n mutual primes, one *odd*, one *even* (in each system). (93)

The general solutions of i, ii, iii, v are

i. $x_2 = m^2+n^2, \quad x_1 = m^2 \sim n^2, \quad y_1 = y_2 = \frac{2}{3}mn$

$x = m^4-\frac{2}{3}m^2n^2+n^4 = (y+x_1^2), \quad y = \frac{4}{3}m^2n^2$

[m, n mutual primes, one odd, one even ; and one a multiple of 3 ; giving x, x_1, x_2 all *odd* ; y, y_1, y_2 all *even*] } . (94, i)

* If these *arithmetical* restrictions be removed, the general solutions here omitted of Systems i, ii, iii, v will be found to merge into the general solutions given of Systems v, iii, ii, i respectively.

ii. $y_1 = m^2+n^2,\ y_2 = m^2\sim n^2,\ x_1 = x_2 = 2mn$

$x = -(3m^4-2m^2n^2+3n^4) = -(y+3y_2^2),\ y = (2mn)^2$

[m, n mutual primes, one odd, one even ; giving x, y_1, y_2 *odd* ; y, x_1, x_2 *even*] (94, ii)

iii. $y_2 = m^2+n^2,\ y_1 = 2mn,\ x_1 = x_2 = m^2\sim n^2$

$x = m^4+10m^2n^2+n^4 = (y+3y_1^2),\ y = (m^2\sim n^2)^2$

[m, n mutual primes, one odd, one even, and one a multiple* of 3 ; giving x, y, x_1, x_2, y_2 *odd* ; y_1 *even*] (94, iii)

v. $x_2 = m^2+n^2,\ x_1 = 2mn,\ y_1 = y_2 = \frac{1}{3}(m^2\sim n^2)$

$x = \frac{1}{3}(m^4+10m^2n^2+n^4) = y+x_1^2,\ y = \frac{1}{3}(m^2\sim n^2)^2$

[m, n mutual primes, one odd, one even ; such that one of $(m \not\sim n)$ is a multiple of 3, giving x, y, y_1, x_2, y_2 *odd* ; x_1 *even*] (94, v)

Examples.—The following Table gives the elements and results of the *lowest* of such numbers $N_1 N_2 = N$ in each of the solvable Cases i, ii, iii, v. The numbers increase so very rapidly that it is not worth continuing. These high numbers (N) would be difficult to factorise otherwise.

Case.	Data.	1st Aurifeuillian.		2nd Aurifeuillian.		Product-Aurifeuillian.	
	m ; n	x_1 ; y_1	N_1	x_2 ; y_2	N_2	x ; y	N
i	3 ; 2	5 ; 4	1729	13 ; 4	22,753	73 ; 48	39,339,937
	3 ; 4	7 ; 8	29857	25 ; 8	307,489	241 ; 192	9,180,699,073
ii	1 ; 2	4 ; 5	4681	4 ; 3	553	− 43 ; 16	2,588,593
	3 ; 2	12 ; 13	204,777	12 ; 5	15,561	−219 ; 144	3,186,534,897
	1 ; 4	8 ; 17	700,297	8 ; 15	416,521	−739 ; 64	
iii	3 ; 2	5 ; 12	176,449	5 ; 13	244,999	457 ; 25	43,229,828,551
iv	1 ; 2	4 ; 1	217	5 ; 1	559	19 ; 3	121,303
	1 ; 4	8 ; 5	4921	17 ; 5	67,471	139 ; 75	332,024,791

29. *Conformal Continued Trin-Aurifeuillian Products.*—The conditions for these are so much more complex than those for the similar problem in Bin-Aurifeuillians that it has not been found possible to extend the above solutions to the product of three Trin-Aurifeuillians.

* Otherwise y, N_2, and N will be multiples of 9.

30. *Quotient Trin-Aurifeuillians.*—Writing 3^{ω}, x^{ω}, y^{ω} for 3, x, y (where $\omega \neq 3\kappa$) in the expression (65), it becomes

$$\mathfrak{X} = \frac{x^{6\omega}+3^{3\omega}y^{6\omega}}{x^{2\omega}+3^{\omega}y^{2\omega}} = x^{4\omega}-3^{\omega}x^{2\omega}y^{2\omega}+3^{2\omega}y^{4\omega} \tag{95}$$

$$= X^4-3X^2Y^2+3^2Y^4, \text{ where } X = x^{\omega},\ Y = 3^{\frac{1}{2}(\omega-1)}y^{\omega}. \tag{95a}$$

Hence $\mathfrak{X}$ is a Trin-Aurifeuillian; and, writing

$$Q = \frac{\mathfrak{X}}{N} = \frac{x^{6\omega}+3^{3\omega}y^{6\omega}}{x^{2\omega}+3^{\omega}y^{2\omega}} \cdot \frac{x^2+3y^2}{x^6+3^3y^6}, \tag{96}$$

it is seen that $\mathfrak{X}$ contains N, and that Q is the A.P.F. of $(x^{6\omega}+3^{3\omega}y^{6\omega})$ when ω is prime.

Hence Q is a Quotient Trin-Aurifeuillian, resolvable into factors S, T; and Q, S, T are expressible in quadratic forms conformal with those of $\mathfrak{X}$, $\mathfrak{L}$, $\mathfrak{M}$, as in Art. **7–7b.**

As the chief interest of these numbers (Q) is for small values of ω, ($= 5$, 7, &c.), it is better to pass on at once to these simple cases, (as general formulæ would be very lengthy).

30a. *Quintic Trin-Aurifeuillian.*—Writing $\omega = 5$ in the formula (95),

$$\mathfrak{Q} = \frac{x^{30}+3^{15}y^{30}}{x^{10}+3^5y^{10}} = x^{20}-3^5x^{10}y^{10}+3^{10}y^{20}. \tag{97}$$

Here $Q = \mathfrak{Q}+N$ can have each of the partitions (A^2+3B^2), (u^2-5v^2), (V^2+15W^2), denoted by Q_3, Q_5, Q_{15}, (accented as required, Art. **5, 21**),

$$Q = \frac{\mathfrak{Q}}{N} = \frac{x^{30}+3^{15}y^{30}}{x^{10}+3^5y^{10}} \cdot \frac{x^2+3y^2}{x^6+3^3y^6} = \frac{x^{20}-3^5x^{10}y^{10}+3^{10}y^{20}}{x^4-3x^2y^2+3^2y^4}, \tag{98}$$

$$= x^{16}+3x^{14}y^2-3^3x^{10}y^6-3^4x^8y^8-3^5x^6y^{10}+3^7x^2y^{14}+3^8y^{16}, \tag{98a}$$

$$Q_3^{\circ} = (x^8-3^2x^4y^4+3^4y^8)^2+3(xy)^2(x^6+3x^4y^2-3^2x^2y^4-3^3y^6)^2, \tag{98b}$$

$$Q_5' = (x^8+8\,.\,3x^6y^2+13\,.\,3^2x^4y^4+8\,.\,3^3x^2y^6+3^4y^8)^2-5(3xy)^2(x^2+3y^2)^6, \tag{98c}$$

$$Q_{15}' = (x^8-2\,.\,3x^6y^2+3\,.\,3^2x^4y^4-2\,.\,3^3x^2y^6+3^4y^8)^2+15(xy)^2(x^6-3x^4y^2+3^2x^2y^4-3^3y^6)^2, \tag{98d}$$

$$S = \mathfrak{L} \div M = (x^{10}-3^3x^5y^5+3^5y^{10}) \div (x^2+3xy+3y^2) \tag{99}$$

$$= x^8-3x^7y+6x^6y^2-3^2x^5y^3+3^2x^4y^4-3^3x^3y^5+2\,.\,3^3x^2y^6-3^4xy^7+3^4y^8. \tag{99a}$$

Here S has three alternative forms (A^2+3B^2) denoted by S_3', S_3'', S_3'''; and *two* alternative forms of (u^2-5v^2)* denoted by S_5', S_5''; and one form (S_{15}) of (W^2+15Z^2).

$$S_3' = (x^4-\tfrac{3}{2}x^3y+\tfrac{3}{2}x^2y^2-\tfrac{9}{2}y^4)^2+3\left(\frac{y}{2}\right)^2(x^3-3x^2y+6xy^2-9y^3)^2, \ (y \text{ even}), \tag{99b}$$

$$S_3'' = \left\{\tfrac{1}{2}x(x^3-3xy^2+9y^3)-9y^4\right\}^2+3\left(\frac{x}{2}\right)^2(x^3-2x^2y+3xy^2-3y^3)^2, \ (x \text{ even}), \tag{99b}$$

$$S_3''' = \tfrac{1}{4}(x^4-3x^3y+6x^2y^2-9xy^3+9y^4)^2+3\,.\,\tfrac{1}{4}(x^4-x^3y+3xy^3-9y^4)^2, \ (x, y \text{ both odd}), \tag{99b}$$

$$S_5' = (x^4-\tfrac{3}{2}x^3y+\tfrac{15}{2}x^2y^2-\tfrac{9}{2}xy^3+9y^4)^2-5\left(\frac{3xy}{2}\right)^2(x^2-xy+3y^2)^2, \ (x \text{ or } y \text{ even}), \tag{99c}$$

$$S_5'' = (\tfrac{3}{2}x^4-6x^3y+15x^2y^2-18xy^3+\tfrac{27}{2}y^4)^2-5(\tfrac{1}{2}x^4-3x^3y+6x^2y^2-9xy^3+\tfrac{9}{2}y^4)^2, \quad (x, y \text{ both odd}), \tag{99c}$$

$$S_{15} = \left\{x^4-\tfrac{3}{2}xy(x^2+3y^2)+9y^4\right\}^2+15\,.\,\tfrac{1}{4}(xy)^2(x^2-3y^2)^2. \tag{99d}$$

The values of $T = \mathfrak{M} \div L$, and of its quadratic partitions T_3', T_3'', T_3'''; T_5', T_5''; T_{15} may be derived from those of S by writing $-y$ for y, (Art. **3**).

* *See* footnote * to Art. **16b.**

The quadratic partitions of Q, viz., Q_3°, Q_3', Q_3'', Q_3'''; Q_5°, Q_5', Q_5''; Q_{15}', Q_{15}'' may be obtained by conformal multiplication of the similarly accented quadratic forms of S, T; each such product yielding two forms of Q, viz., Q° and *one* of Q', Q'', Q'''.

31. *Fermat's Numbers.* (Base 3). — These are numbers of form $\mathfrak{F}_n = 3^n+1$. Writing $x=1$, $y=3^\nu$ (ν any integer), the Trin-Aurifeuillian N (Art. **21**) becomes

$$N = (1+3^3.3^{6\nu}) \div (1+3.3^{2\nu}) = (1+3^{6\nu+3}) \div (1+3^{2\nu+1}). \quad (100)$$

Hence the A.P.F. of all Fermat's Numbers $\mathfrak{F}_n = 3^n+1$, where $n = 6\nu+3$, is resolvable into the two factors L, M of Art. **21**, viz.,

$$L = 3^{2\nu+1}-3^{\nu+1}+1, \quad M = 3^{2\nu+1}+3^{\nu+1}+1, \quad (100a)$$

and these (L, M) can be at once *completely* resolved into their *prime* factors as far as $n = 39$.* Also the A.P.F. of $\mathfrak{F}_{45} = (3^{45}+1)$ is *completely* resolvable by the formulæ of Art. **30a**. Beyond this limit the factorisation into *prime* factors requires the aid of "special† divisors." When $\frac{1}{3}n = 2\nu+1$ is *composite*, the *Reduced A.P.F.* of $\mathfrak{F}_n$ is similarly resolvable into two factors (S, T) by Art. **7b**.

Examples.—The following Table shows the *complete* resolution‡ of the *Reduced A.P.F.* of these numbers into their *prime* factors, all but complete§ up to $n = 105$: the factorisation of the Aurifeuillian Factors L, M or S, T being shown separately: prime factors only are shown; the semicolon (;) denotes that the factorisation is *complete*.

[*Note* that the highest number $(3^{105}+1)$, here *completely* factorised, contains 51 figures.].

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* At this limit the *larger* Aurifeuillian Factor M is > 9 million, the present limit of the large Factor-Tables.

† Reuschle's *Neue zahlentheoretische Tabellen*, above quoted, gives (pp. 42–46) a table of "special prime divisors" (p) of (3^n-1) up to $p < 1000$.

‡ The whole of this Table has been worked out by the author for this paper, though much of it had been previously obtained by others. The initials B, C, L (in the columns of L or S, M or T) indicate the *original* computer (so far as known to the author) who first succeeded in *completely* factorising the Aurifeuillian Factors into their *prime* factors, viz.,

B denotes Mr. C. E. Bickmore; L denotes M. Le Lasseur;
C denotes Lt.-Col. A. Cunningham, (the present author).

(1) Mr. Bickmore gives the results when $n = 3, 9, 15, 21, 27$; 87, 105 in his first paper quoted, pp. 8, 43; (except that, when $n = 105$, S is not factorised). The remaining results marked B in the Table were communicated to the author by Mr. Bickmore in MS: besides this, Mr. Bickmore has considerably aided in, and to some extent verified, the author's results marked C.

(2) The results when $n = 87$ are ascribed to M. Le Lasseur in Ed. Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 12.

(3) Most of the factorisations and detection of high primes done by the author (C) were effected by the method developed in Mr. Lawrence's Paper (see footnote § to Art. **18**).

§ Complete except that the factorisation of the smaller Aurifeuillian Factor L, when $n = 31$, is undetermined.

FACTORS OF THE REDUCED A.P.F. OF (3^n+1). $[n = 6\nu+3]$.

n	L or S		M or T	
3	1 ;		7 ;	
9	19 ;		37 ;	
15	31 ;		271 ;	
21	7 · 43 ;		2269 ;	
27	19,441 ;	B	19927 ;	B
33	176,419 ;	B	25411 ;	B
39	79 · 2887 ;	B	157 · 1041 ;	B
45	387,631 ;	B	755,551 ;	B
51	613 · 30,091 ;	B	129,159,847 ;	B, C
57	3079 · 53,923 ;	B	1,162,320,517 ;	B, C
63	127 · 883 · 2521 ;	B	550,554,229 ;	C
69	139 · 5107 · 132619 ;	C	9109 · 1,476,463 ;	C
75	26,251 · 119,101 ;	B	2551 · 1,530,601 ;	C
81	163 · 208,657 · 224,209 ;	C	1297 · 5,879,415,781 ;	C
87	261,697 · 37,464,463 ;	L, B	349 · 142,159 · 1,383,301 ;	L
93			373 · 541,447 · 3,058,399 ;	C
99	199 · 4357 · 337,448,233 ;	C	397 · 378,450,588,583 ;	C
105	24151 · 3,369,031 ;	C	211 · 1051 · 3,454,081 ;	B
111	223 ·			
117	937 ·		1171 ·	

Note.—To complete the factorisation of these numbers $\mathfrak{N}_n = (3^n+1)$, the factorisation of the *algebraic factors* of form $F = 3^\omega+1$, where $\omega \not\equiv 31$, is required. This factorisation, when $\omega < 19$, is given in Mr. Bickmore's first paper, above quoted, p. 43. The remaining results, still required, are*

$\frac{1}{4}(3^{19}+1)$
$= 2851 \cdot 101,917$;
$\frac{1}{4}(3^{23}+1)$
$= 23,535,794,707$;
$\frac{1}{4}(3^{31}+1)$
$= 6883$
$\cdot 22,434,744,889$.

32. *Fermat's Numbers.* (Higher Bases).—Writing in Formulæ (65) $X = 1$, $Y = 3y^3$, $n = 3\omega = 6\nu+3$, (ν being any integer), then

$$\mathfrak{A} = \frac{1+Y^{3\omega}}{1+Y^{\omega}} = \frac{1+(3y^3)^{6\nu+3}}{1+(3y^3)^{2\nu+1}} = \frac{1+3^3.\eta^6}{1+3.\eta^2}, \quad \text{where } \eta = 3^\nu.y^{2\nu+1}. \quad (101)$$

Here $\mathfrak{A}$ is a *Trin-Aurifeuillian*, so that every such number, and also its A.P.F., viz.,

$\mathfrak{A} \div N = \{(1+Y^{3\omega})(1+Y)\} \div \{(1+Y^{\omega})(1+Y^3)\}$, where $Y = 3y^2$;

and also its "Reduced A.P.F.," (when further reducible), admit of resolution into the two factors L, M, or S, T of Art. 7a, b, 30.

The existing Factor-Tables (extending to 9 million) enable the *complete* factorisation (*i.e.*, into prime factors) of such numbers to be effected up to the following limits :—

$3\omega =$	3 ,	9 ,	15 ,	21 ,	27 to 45
$Y = 3y^2 =$	$3 \cdot 1731^2$†,	$3 \cdot 8^2$,	$3 \cdot 4^2$,	$3 \cdot 2^2$,	$3 \cdot 1^2$
	= 8,989,083,	192 ,	48 ,	12 ,	3

and, by the aid of "special divisors,"‡§ it can be carried far beyond these limits.

Examples.—The following table shows the resolution‖ into *prime* factors of

* The first result is due to Mr. Bickmore ; the other two are due to the author.

† Reference note (†) at foot of p. 430.

‡ Reuschle's *Tafeln complexer Primzahlen* give (p. 193 *et seq.*) the *least* values of Y for which $(1+Y^{3\omega})$ is divisible by any *given* prime $p < 1000$.

§ See footnote † to Art. 18 about the author's MS. Tables of these.

‖ The whole of this Table has been worked out by the author for this paper.

B, C indicate the *original* computer, as in footnote ‡ to Art. 31, *q.v.*

N = A.P.F. of $(1 + Y^n)$, where $Y = 3y^2$ and $n = 3\omega$ to various high limits, for the index n = 3ω 3†, 9, 15, 21, 33.

The factorisation of the Aurifeuillian Factors L, M or S, T is shown separately, and *prime* factors only are shown the semicolon (;) where $n = 9, 15, 21, 27, 33$ indicates that the factorisation is *complete*.

FACTORISATION OF THE A.P.F. OF $(1 + Y^{3\omega})$, [where $Y = 3y^2$].

y	$Y = 3y^2$	$(1+Y^3)+(1+Y^1)$		$(1+Y^9)+(1+Y^3)$		$\{(1+Y^{15})(1+Y)\}+\{(1+Y^5)(1+Y^3)\}$		
		L	M	L	M	L or S		M or T	
1	3	1	7	19 ;	37 ;	31 ;		271 ;	
2	12	7	19	1657 ;	1801 ;	31 . 421 ;	B	35,671 ;	B
3	27	19	37	19,241 ;	19,927 ;	387,631 ;	B	755,551 ;	B
4	48	37	61	110,017 ;	19 . 5851 ;	31 . 134,731 ;	C	6,887,341 ;	C
5	75	61	7 . 13	127 . 3313 ;	423,001 ;	331 . 78,781 ;	C	31 . 151 . 8311 ;	C
6	108	7 . 13	127	181 . 6949 ;	19 . 66403 ;	115,693,471 ;	C	161,466,931 ;	C
7	147	127	13 . 13	19 . 167,023 ;	3,179,611 ;	151 . 241 . 11,161 ;	C	12,511 . 43,201 ;	C
8	192	13 . 13	7 . 31	7,073,281 ;	19 . 372,763 ;	1,202,388,841 ;	C	211 . 7,317,091 ;	C
9	243	7 . 13	271	37 . 387631 ;	19 . 755,551 ;	26,251 . 119,101 ;	B	2551 . 1,530,601 ;	C
10	300	271	331	19 . 487 . 2917 ;	37 . 109 . 181 ;	151 . 4231 . 11,491 ;	C	31 . 541 . 534,661 ;	C
11	363	331	397	19 . 37 . 68,023 ;	73 . 655,399 ;	31 . 512,130,481 ;	C	241 . 79,011,151 ;	C
12	432	397	7 . 67	37 . 2,178,541 ;	19 . 4,244,059 ;			61 . 61 . 211 . 48,271 ;	C
13	507	7 . 67	547	19 . 6,858,109 ;	73 . 109 . 16,381 ;	31 . 61 . 32,386,261 ;	C		
14	588	547	631		109 . 1,865,341 ;	61 . 5821 . 313,711 ;	C	151 . 9721 . 87,541 ;	C
15	675	631	7 . 103	19 .	73 . 163 . 25,849 ;	61 . 751 . 4,242,421 ;	C	31 .	
16	768	7 . 103	19 . 43	37 .	937 . 483,481 ;	31 . 61 . 172,938,811 ;	C		
17	867	19 . 43	919	127 . 577 . 8893 ;		2221 . 12,841 . 18,691 ;	C		
18	972	919	13 . 79			541 . 1561,577,671 ;	C	31 . 151 . 181 . 211 . 5281 ;	C

[*Note* that the highest number $(1 + Y^{15})$, here *completely* factorised, contains 45 figures: the factorisation of $(1 + Y^5)$ is required to render these really complete; it is hoped to publish these hereafter.]

† Reference note (†) at foot of p. 430.

FACTORISATION OF THE A.P.F. OF $(1+Y^{3w})$, [where $Y=3y^2$].—*Continued.*

y	$Y=3y^2$	$\{(1+Y^{21})(1+Y)\}\div\{(1+Y^7)(1+Y^3)\}$			
		L or S		M or T	
1	3	7 · 43 ;		2269 ;	
2	12	1,885,339 ;	B	7 · 43 · 17,011 ;	B
3	27	127 · 883 · 2521 ;	B	550,554,229 ;	C
4	48	43 ·			
5	75	883 ·			
6	108				
7	147	43 ·			
8	192	421 ·		757 ·	
9	243	2269 · 24,151 · 3,369,031 ;	C	7 · 43 · 211 · 1051 · 3,454,081 ;	B
10	300			463 ·	

y	$Y=3y^2$	$(1+Y^{27})\div(1+Y^9)$			
		L		M	
1	3	19,441 ;	B	19,927 ;	B
2	12	109 · 47,336,293 ;	B	271 · 487 · 39,037 ;	B
3	27	163 · 208,657 · 224,209 ;	C	1297 · 5,879,415,781 ;	C

y	$Y=3y^2$	$\{(1+Y^{33})(1+Y)\}\div\{(1+Y^{11})(1+Y^3)\}$			
		L or S		M or T	
1	3	176,419 ;	B	25,411 ;	B
2	12	67 · 199 · 1453 · 5479 ;	B		
3	27	199 · 4357 · 337,448,233 ;	C	397 · 378,450,588,583 ;	C

† Refer to †, pp. 428, 429. The highest number of this kind, $N=(1+Y^3)\div(1+Y)$, factorisable by the existing Factor Tables is

$$(1+8{,}989{,}083^3)\div(1+8{,}989{,}083) = (7\cdot 73\cdot 17581)\cdot(19\cdot 473{,}383).$$

It will be noticed that these numbers form a *Trin-Aurifeuillian Chain* (Art. **34**), *i.e.*, the M of each number = the L of the next higher number.

33. *Common Aurifeuillian Factor of Trin-Aurifeuillians.*—It will now be shown that every number F of form (A^2+3B^2) is an Aurifeuillian Factor of *six* different Trin-Aurifeuillians.

Take N_r as the type of six different Trin-Aurifeuillians, such that

$$N_r = (x_r^6+3^3y_r^6) \div (x_r^2+3y_r^2) = L_rM_r, \quad [r = 1, 2, 3, 4, 5, 6]. \quad (102)$$

Then, taking $N_1<N_2$, $N_3<N_4$, $N_5<N_6$, and $L_r<M_r$ in general, F will be a common factor of all the six Ns, when

$$F = M_1 = L_2 = M_3 = L_4 = M_5 = L_6. \quad (103)$$

Next expressing the three Ls and three Ms by pairs in the three quadratic forms $(A_r^2+3B_r^2)$ of Art. **21**,

$$F = A^2+3B^2 = M_1' = L_2' = M_3'' = L_4'' = M_5''' = L_6'''. \quad (103a)$$

Equating each quadratic part (A_r, B_r) of the Ls and Ms separately to the parts (A, B) of F in turn, these give the relations

$$\left.\begin{aligned} A &= x_1+\tfrac{3}{2}y_1 = x_2\sim\tfrac{3}{2}y_2 = \tfrac{1}{2}x_3 = \tfrac{1}{2}x_4 = \tfrac{1}{2}(x_5+3y_5) = \tfrac{1}{2}(x_6\sim 3y_6) \\ B &= \tfrac{1}{2}y_1 = \tfrac{1}{2}y_2 = \tfrac{1}{2}x_3+y_3 = \tfrac{1}{2}x_4\sim y_4 = \tfrac{1}{2}(x_5+y_5) = \tfrac{1}{2}(x_6\sim y_6) \end{aligned}\right\} \quad (104)$$

whence, finally,

$$\left.\begin{aligned} x_1 &= A-3B, & y_1 &= 2B; \\ x_2 &= A+3B, & y_2 &= 2B; \\ x_3 &= 2A, & y_3 &= B\sim A; \\ x_4 &= 2A, & y_4 &= B+A; \\ x_5 &= 3B\sim A, & y_5 &= A\sim B; \\ x_6 &= 3B+A, & y_6 &= A+B; \\ L_1' &= (A\sim 6B)^2+3B^2, & M_1' &= A^2+3B^2 \\ L_2' &= A^2+3B^2, & M_2' &= (A+6B)^2+3B^2 \\ L_3'' &= A^2+3\,(2A\sim B)^2, & M_3'' &= A^2+3B^2 \\ L_4'' &= A^2+3B^2, & M_4'' &= A^2+3\,(2A+B)^2 \\ L_5''' &= (3B\sim 2A)^2+3\,(2B\sim A)^2, & M_5''' &= A^2+3B^2 \\ L_6''' &= A^2+3B^2, & M_6''' &= (3B+2A)^2+3\,(A+2B)^2 \end{aligned}\right\} \quad (105)$$

Hence many simple relations may be found between such a set of

six Trin-Aurifeuillians, *e.g.*,

$$\left.\begin{aligned} N_1+N_4 = N_2+N_3 = N_5+N_6 = 14\,(A^2+3B^2) = 14F^2 \\ N_2-N_1 = N_4-N_3 = \tfrac{1}{2}\,(N_6-N_5) = 24ABF \end{aligned}\right\}. \quad (106)$$

In order that the six Trin-Aurifeuillians should fulfil the limitations of (65), it suffices that A, B should be one *odd*, one *even*, and A prime to $3B$; these make each x prime to its own $3y$, and N *odd*.

The above theorem is true whether F be prime, or composite; but, if $F = A^2+3B^2$ be a *composite*, it may be considerably extended. For every factor of F must in this case be of form $(\alpha^2+3\beta^2)$, and F will then be expressible in several ways in form (A^2+3B^2), and the above theorem will be independently true for *each* of these different modes of expression of F.

Again, since every Trin-Aurifeuillian (N) is the product of *two* factors (L, M), each of which is of form (A^2+3B^2), it follows from the above that every Trin-Aurifeuillian (N) is an Aurifeuillian factor of 12 different Trin-Aurifeuillians.

Examples.—The following Tables show the 6 Trin-Aurifeuillians which contain the common Aurifeuillian Factor $F=7$, or 13, and the set of 12 Trin-Aurifeuillians which contain the Trin-Aurifeuillian $F = 133$.

Data.			Trin-Aurifeuillians.					
F	A, B		N_1	N_2	N_3	N_4	N_5	N_6
7	2, 1	x, y =	−1, 2	5, 2	−4, 1	4, 3	1, 1	5, 3
		L, M =	19 . 7	7 . 67	31 . 7	7 . 79	1 . 7	97 . 7
		N =	133	469	217	553	7	679
13	1, 2	x, y =	−5, 4	7, 4	2, 1	2, 3	5, 1	7, 3
		L, M =	133 . 13	13 . 181	1 . 13	13 . 49	43 . 13	13 . 139
		N =	1729	2353	13	637	559	1807
133	11, 2	x, y =	−5, 4	17, 4	22, 9	22, 13	−5, 9	17, 13
		L, M =	13 . 133	133 . 541	1321 . 133	133 . 1849	403 . 133	133 . 1459
		N =	1729	71953	175693	245917	53599	194047
	5, 6	x, y =	13, 21	23, 12	10, 1	10, 11	13, 1	23, 11
		L, M =	1069 . 133	133 . 1789	73 . 133	133 . 793	211 . 133	133 . 1651
		N =	142177	237937	9709	105469	28063	199583

34. ***Trin-Aurifeuillian Chains.***—Let a series of Trin-Aurifeuillians, $N_1, N_2, N_3, \ldots, N_r$ be taken, so that

$$\ldots\ldots\ldots

and let the parts (x_r, y_r) of the successive functions (N_r) be taken, so that

Case i. $y_1 = y_2 = \ldots = y_r$, (an *even* number),

$x_2 = x_1 + 3y_1$, $x_3 = x_2 + 3y_2$, ..., $x_r = x_{r-1} + 3y_{r-1}$, (all *odd*). (108, i)

Case ii. $x_1 = x_2 = \ldots = x_r$, (an *even* number),

$y_2 = y_1 + x_1$, $y_3 = y_2 + x_2$, ..., $y_r = y_{r-1} + x_{r-1}$, (all *odd*). (108, ii)

Case iii. $x_1^2 - 3y_1^2 = x_2^2 - 3y_2^2 = \ldots = x_r^2 - 3y_r^2 = z^2$, (an *odd* square),

where x_r, y_r are one *odd*, one *even*. (108, iii)

Now, by the formulæ for L', M'; L'', M'', Art. **21**,

Case i. $M'_{r-1} = (x_{r-1} + \frac{3}{2}y_{r-1})^2 + 3(\frac{1}{2}y_{r-1})^2$, $L'_r = (x_r - \frac{3}{2}y_r)^2 + 3(\frac{1}{2}y_r)^2$,

Case ii. $M''_{r-1} = (\frac{1}{2}x_{r-1})^2 + 3(\frac{1}{2}x_{r-1} + y_{r-1})^2$, $L''_r = (\frac{1}{2}x_r)^2 + 3(\frac{1}{2}x_r \sim y_r)^2$.

Case iii. See next Article.

Hence, since $L = L' = L''$, and $M = M' = M''$, (the accents merely indicating the quadratic partition),

Cases i, ii, iii. $M_{r-1} = L_r$, (in each case), (109)

or, in other words, every *successive* pair N_{r-1}, N_r have a common Aurifeuillian Factor. Such a series may be called a *Trin-Aurifeuillian Chain*, and has many interesting properties. Thus

$$N_r : N_{r-1} = L_r . M_r : L_{r-1} . M_{r-1} = M_r : L_{r-1}. \quad (110a)$$

Similarly, $N_{r-2} : N_{r-3} = M_{r-2} : L_{r-3}$.

Hence in every *successive* quartet $(N_{r-3}, N_{r-2}, N_{r-1}, N_r)$

$$N_r . N_{r-2} : N_{r-1} . N_{r-3} = M_r : L_{r-3}. \quad (110b)$$

Similarly, giving to r every integer value from 1 to $2r$,

$$\frac{N_{2r} . N_{2r-2} \ldots N_6 . N_4 . N_2}{N_{2r-1} . N_{2r-3} \ldots N_5 . N_3 . N_1} = \frac{M_{2r}}{L_1}. \quad (110c)$$

34a. *Trin-Aurifeuillian Chains.* (Case iii).—This case requires special study. It is defined by

$$x_r^2 - 3y_r^2 = z^2 \quad (\text{an } \textit{odd} \text{ square}), \quad [r = 1, 2, 3, \&c.]. \quad (111)$$

Here z must be either an *odd prime* of form $z = 12\zeta \pm 1$, or a product

of such primes. Now, for any given *prime** value of z of this form >1, there are *two minimum solutions*, say (x_1, y_1), (ξ_1, η_1), connected by the reciprocal relations

$$\left.\begin{array}{l} x_1 = 2\xi_1 \sim 3\eta_1,\ y_1 = \xi_1 \sim 2\eta_1;\quad \xi_1 = 2x_1 \sim 3y_1,\ \eta_1 = x_1 \sim 2y_1 \\ \text{one of } x_1, \xi_1 \textit{ odd}, \text{ one } \textit{even};\ y_1, \eta_1 \text{ of opposite affection to } x_1, \xi_1 \end{array}\right\}. \quad (112a)$$

Also each such "base-solution"* gives rise to an infinite series of solutions, the *successive* solutions of either series being connected by the similar relations,

$$x_r = 2x_{r-1} + 3y_{r-1},\ y_r = x_{r-1} + 2y_{r-1};\ \xi_r = 2\xi_{r-1} + 3\eta_{r-1},\ \eta_r = \xi_{r-1} + 2\eta_{r-1}, \quad (112b)$$

$$x_{r-1} = 2x_r - 3y_r,\ -y_{r-1} = x_r - 2y_r;\ \xi_{r-1} = 2\xi_r - 3\eta_r,\ -\eta_{r-1} = \xi_r - 2\eta_r. \quad (112c)$$

The last two relations are similar in form to, but not identical with, Equation (112*a*); such a pair of series will be called *Conjugate Series.*

Now let $$N_r = x_r^4 - 3x_r^2y_r^2 + 9y_r^4 = L_r . M_r,$$

$$\overline{N}_r = \xi_r^4 - 3\xi_r^2\eta_r^2 + 9\eta_r^4 = \overline{L}_r . \overline{M}_r,$$

be the Trin-Aurifeuillians formed from (x_r, y_r), (ξ_r, η_r) respectively.

Also, for distinctness, suppose

$$x_1 \textit{ odd};\ \text{then } x_1, \eta_1 \text{ are both } \textit{odd};\ y_1, \xi_1 \textit{ even};$$

and, by Art. 21,

$$L_1 = (x_1 \sim \tfrac{3}{2}y_1)^2 + 3(\tfrac{1}{2}y_1)^2,\quad \overline{L}_1 = (\tfrac{1}{2}\xi_1)^2 + 3(\tfrac{1}{2}\xi_1 \sim \eta_1)^2,$$

and, the relations (112*a*) now give

$$L_1 = \overline{L}_1, \quad (113)$$

so that the lowest Trin-Aurifeuillians N_1, $\overline{N}_1$ of two conjugate series have the *same* smaller Aurifeuillian Factor (*i.e.*, $L_1 = \overline{L}_1$).

Again, the assumption x_1 *odd* involves

$$x_1, y_2, x_3, \ldots, x_{2r-1}, y_{2r}, x_{2r+1}, \text{ all } \textit{odd},$$
$$y_1, x_2, y_3, \ldots, y_{2r-1}, x_{2r}, y_{2r+1}, \text{ all } \textit{even}.$$

Hence, by Art. 21,

$$M'_{2r-1} = (x_{2r-1} + \tfrac{3}{2}y_{2r-1})^2 + 3(\tfrac{1}{2}y_{2r-1})^2;\quad L''_{2r} = (\tfrac{1}{2}x_{2r})^2 + 3(\tfrac{1}{2}x_{2r} \sim y_{2r})^2,$$
$$M''_{2r} = (\tfrac{1}{2}x_{2r})^2 + 3(\tfrac{1}{2}x_{2r} + y_{2r})^2;\qquad L'_{2r+1} = (x_{2r+1} \sim \tfrac{3}{2}y_{2r+1})^2 + 3(\tfrac{1}{2}y_{2r+1})^2;$$

* When z is *composite*, these "base-solutions" are more numerous: they are connected in pairs by the reciprocal relation (112*a*); and each base-solution gives rise to its own *Trin-Aurifeuillian Chain* as when z is *prime*. *Example.*—When $z = 11.13$, there are *four* base solutions; see Table below.

and the relations (112*b*, *c*) now give

$$M'_{2r+1} = L''_{2r}, \quad M''_{2r} = L'_{2r+1},$$

which are equivalent to $M_{r-1} = L_r$ (for all values of r). (114)

By similar reasoning, in the conjugate $(\xi_r, \eta_r, \overline{N}_r)$ series, $\overline{M}_{r-1} = \overline{L}_r$. Hence these series are *Trin-Aurifeuillian Chains*, as defined in last Article (Case iii). The property that each number (N_r) of these series is expressible in the *three* forms $N = N` = N_{vi}$ has been already treated of in Art. **25, 26**

From the above it is seen that any such Trin-Aurifeuillian N_1 gives rise to *two* chains, in each of which it is the lowest number, viz., one of Case iii and one of Case i or Case ii according as y_1 or x_1 is even. Also any number L_1 of form (A^2+3B^2) gives rise to *four* chains in which it is a common Aurifeuillian Factor (L_1) of the lowest number (N_1) of each chain, viz., *one* of Case i, *one* of Case ii, and *two* of Case iii.

Examples.—The table shows the quadratic parts (x_r, y_r) of a few numbers N_1, N_2, &c. of the Trin-Aurifeuillian Chains arising when $z = 11, 13$, and 143 $(= 11.13)$. The conjugate chains are in each case bracketed together. Each number N_r can be expressed in the three forms N, $N`$, N_{vi} by the formulæ of Art. **26** from the given x, y, z. The important case when $z = 1$ has been given in Art. **26** in fuller detail.

z	N_1	N_2	N_3	N_4
	x_1 ; y_1	x_2 ; y_2	x_3 ; y_3	x_4 ; y_4
11	13 ; 4	38 ; 21	139 ; 80	518 ; 299
	14 ; 5	43 ; 24	158 ; 91	589 ; 340
13	14 ; 3	37 ; 20	134 ; 77	499 ; 288
	19 ; 8	62 ; 35	229 ; 132	854 ; 493
143	146 ; 17	343 ; 180	1226 ; 703	&c.
	241 ; 112	818 ; 465	3031 ; 1748	&c.
	151 ; 28	386 ; 207	1393 ; 800	&c.
	218 ; 95	721 ; 408	2666 ; 1537	&c.

35. *Bin- and Trin-Aurifeuillians.*—These two functions have many properties in common, which are worth study together. A summary is given below. [Here C.F. denotes *Common Factor.*]

ART.	*Bin-Aurifeuillians*, N, $\mathfrak{N}$.	ART.	*Trin-Aurifeuillians*, N, $\mathfrak{N}$.
9.	Dimorphs are possible.	**24.**	Dimorphs are possible.
10.	$N = \mathfrak{U}$ or $\mathfrak{M}$ of $\mathfrak{N}$, when $x^2 \mp 2y^2$, or $2y^2 - x^2 = z^2$.	**26.**	$N = \mathfrak{U}$ of $\mathfrak{N}$, when $x^2 - 3y^2 = z^2$.
11.	$N = X^4 + Y^4$ is impossible.	**26.**	$N = (x^6 + z^6) + (x^2 + z^2)$, when x or $2x = m^2 + 3n^2$.
11.	$N = X^4 - Y^4$ is possible.		
18a.	$N_1 N_2 = N$, when $x_2^2 \mp x_1^2 = (2y_1)^2 = (2y_2)^2$; or $y_2^2 \mp y_1^2 = x_1^2 = x_2^2$.	**28a.**	$N_1 N_2 = N$, when $x_2^2 \sim x_1^2 = y_1^2 = y_2^2$, or $y_2^2 \sim y_1^2 = x_1^2 = x_2^2$.
17.	Factorisation of $(2^{4\nu+2}+1)$.	**31.**	Factorisation of $(3^{6\nu+3}+1)$.
18.	Factorisation of $(2y^2)^{4\nu+2}+1$.	**32.**	Factorisation of $(3y^2)^{6\nu+3}+1$.
19.	$F = a^2 + b^2$ is C.F. of 4 numbers N, $N_1 + N_4 = N_2 + N_3 = 6F^2$. Also N is C.F. of 8 numbers $\mathfrak{N}$.	**33.**	$F = A^2 + 3B^2$ is C.F. of 6 numbers N, $N_1 + N_4 = N_2 + N_3 = N_4 + N_5 = 14F^2$. Also N is C.F. of 12 numbers $\mathfrak{N}$.
20.	Bin-Aurifeuillian Chain $(M_{r-1} = L_r)$, when i. $y_r = y_1$, and $x_r = x_{r-1} + 2y_1$, or ii. $x_r = x_1$, and $y_r = x_1 + y_{r-1}$.	**34.**	Trin-Aurifeuillian Chain $(M_{r-1} = L_r)$, when i. $y_r = y_1 = \epsilon$, and $x_r = x_{r-1} + 3y_1$, or ii. $x_r = x_1 = \epsilon$, and $y_r = x_1 + y_{r-1}$.

36. *Bi-Trin-Aurifeuillians.*—The question now arises as to whether any numbers (N) exist, expressible at once in the Bin-Aurifeuillian (N_2), Trin-Aurifeuillian (N_3), and Trin-Ant-Aurifeuillian (N_3') forms.

A *general* solution seems very difficult to obtain; but an interesting partial solution is obtainable by Art. **26**; from which it is seen that, writing

$$N_{\text{vi}} = \frac{x^6 + z^6}{x^2 + z^2}, \quad N_3 = \frac{x^6 + 3^3 y^6}{x^2 + 3y^2}, \quad N_3' = \frac{z^6 - 3^3 y^6}{z^2 - 3y^2}, \tag{115}$$

then the relation

$$x^2 - 3y^2 = z^2 \tag{116}$$

suffices that the number N_{vi} should be expressible as both N_3 and N_3'.

Next, taking

$$z = \zeta^2, \quad x = 2\xi^2, \tag{116a}$$

the function $N_{\text{vi}} = \dfrac{\zeta^{12} + 2^6 \xi^{12}}{\zeta^4 + 2^2 \xi^4}$, a cubic Bin-Aurifeuillian, (say N_2''').

The relation (116) thus becomes

$$4\xi^4 - \zeta^4 = 3y^2. \qquad (116b)$$

Hence, every solution of this Diophantine gives a number expressible in the *four* ways

$$N_{VI} = N_3 = N_3' = N_2'''. \qquad (117)$$

A *general* solution of (116*b*) seems difficult; but, if ξ, ζ, y be any *given* solution, then a *second* solution is given* by the following formulæ:—

$$\left.\begin{aligned} \xi_1 &= \xi(27y^4 \sim 64\xi^4), \quad \zeta_1 = \zeta(27y^4 \sim 4\zeta^8) \\ y_1 &= y\{324y^8 \sim 3(4\xi^4+\zeta^4)^4\} \end{aligned}\right\}. \qquad (118)$$

Similarly a *third* solution (ξ_2, ζ_2, y_2) may be derived; and, repeating the process, an infinity of solutions may be derived from any given solution (ξ, ζ, y).

Examples.—One solution of (116*b*) is obvious, viz., $\xi = 1$, $\zeta = 1$, $y = 1$; giving $z = 1$, $x = 2$, whence

$$N = 13 = \frac{2^6+1^6}{2^2+1^2} = \frac{2^6+3^3.1^6}{2^2+3.1^2} = \frac{1^6 \sim 3^3.1^6}{1^2 \sim 3.1^2} = \frac{1^{12}+2^6.1^{12}}{1^4+2^2.1^4}.$$

Then (118) give a *second* solution, $\xi_1 = 37$, $\zeta_1 = 23$, $y_1 = 1551$; giving $z = 529$ $x = 2738$, whence

$$N = \frac{2738^6+529^6}{2738^2+529^2} = \frac{2738^6+3^3.1551^6}{2738^2+3.1551^2} = \frac{529^6 \sim 3^3.1551^6}{529^2 \sim 3.1551^2} = \frac{23^{12}+2^6.37^{12}}{23^4+2^2.37^4}.$$

The next solution (ξ_2, ζ_2, y_2) is in too high numbers to be practically useful.

36a. *Factorisation of above numbers.*—These numbers, being Aurifeuillians of orders 2 and 3, admit of resolution into Aurifeuillian Factors *in each way;* this greatly helps their ultimate resolution into *prime* factors.

Thus $N = N_2''' = S_2 . T_2$, (Art. **16a**); and $N = N_3 = L_3 M_3$, (Art. **21**).

Ex.—Taking the above high number $N = N_2''' = N_3$,

$$N_2''' = \frac{23^{12}+2^6.37^{12}}{23^4+2^2.37^4} = S_2 T_2 = 3{,}664{,}453 \,.\, 14{,}785{,}321.$$

$$N_3 = \frac{2738^6+3^3.1551^6}{2738^2+3.1551^2} = L_3 M_3 = 1{,}973{,}533 \,.\, 27{,}453{,}361.$$

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* See M. Desboves's "Mémoire sur la résolution en nombres entiers de l'équation $aX^m + bY^m = cZ^m$," Art. 16, Results (67); published in *Nouv. Ann. de Mathém.*, 2e Sér., t. XVIII, 1879.

Here T_2, M_3 are both beyond the range of the Factor-Tables; but the resolution of S_2, L_3 supplies the small factors of M_3, T_2 respectively; thus

$$N = N_2''' = (13 \cdot 61 \cdot 4621)(457 \cdot 32{,}353) = N_3 = (61 \cdot 32{,}353)(13 \cdot 457 \cdot 4621).$$

This large number (14 figures) could not have been otherwise easily factorised.

N.B.—It is hoped to publish later similar investigations on Aurifeuillians of higher orders, *e.g.*, order 5, 6, &c.

APPENDIX.

37. *High Primes.*—The following high primes* occurring in this paper (see Tables, Art. **18, 31, 32**) seem worth separate record. The initials (B, C) indicate the name of the original discoverer, as explained in the footnote ‡ to Art. **31**.

Prime	Factor of	Initial	Prime	Factor of	Initial
9,987,121	$3200^6 + 1$	C	337,448,233	$3^{99} + 1$	C
			512,130,481	$363^{15} + 1$	C
31,720,981	$72^{10} + 1$	C	550,554,229	$3^{63} + 1$	C
32,386,261	$507^{15} + 1$	C	581,433,521	$578^{10} + 1$	C
40,768,781	$450^{10} + 1$	C			
43,120,981	$200^{10} + 1$	C	1,162,320,517	$3^{57} + 1$	B, C
47,336,293	$12^{27} + 1$	B	1,202,388,841	$192^{15} + 1$	C
69,772,181	$722^{10} + 1$	C	1,447,960,181	$200^{10} + 1$	C
79,011,151	$363^{15} + 1$	C	1,561,577,671	$972^{15} + 1$	C
			5,879,415,781	$3^{81} + 1$	C
115,693,471	$108^{15} + 1$	C			
129,159,847	$3^{51} + 1$	B, C	22,434,744,889	$3^{31} + 1$	C
161,466,931	$108^{15} + 1$	C	23,535,794,707	$3^{23} + 1$	C
172,938,811	$768^{15} + 1$	C			
253,765,901	$882^{10} + 1$	C	378,450,588,583	$3^{99} + 1$	C

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* Believed to be all new. The mode of detection of these primes is explained in the footnotes †, ¶ to Art. **18** and ‡ (3) to Art. **31**.

The Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups. *By* E. B. ELLIOTT.

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1. On a number of previous occasions, and more particularly in a paper "On the Interchange of the Variables in certain Linear Differential Operators" [*Phil. Trans.*, Vol. CLXXXI. (1890), A, pp. 19–51], I have applied a simple but artificial method to the transformation of linear differential operators from forms in which their arguments are the successive derivatives of a dependent with regard to one or more independent variables to the forms which they assume when dependent and independent variables are transposed. Again, in a paper "On the Reversion of Partial Differential Expressions with Two Independent and Two Dependent Variables" (*Proc. Lond. Math. Soc.*, Vol. XXII., pp. 79–104), I have somewhat extended the method, by application to a case in which there are more independent variables than one. Up to the present time I have not, nor, so far as I know, has any one else, applied the method to the transformation of operators consequent on continuous transformations of, or substitutions for, dependent and independent variables in terms of new dependent and independent variables; but it is one which admits of wide applications in this direction. The following paper deals with a class of such applications.

I.

2. There is no new difficulty of principle when the formulæ of transformation are linear, and the original variables are supposed connected by one relation only, so that a single one of them is dependent and the rest independent. Let us first consider the case of two variables x, y, supposed connected by one relation of quite arbitrary form and not necessarily known, and consider them, and the successive derivatives of the latter with regard to the former, to be expressed in terms of x', y', and the successive derivatives of y' with regard to x' by the transformations of the general linear group

$$\left.\begin{aligned} x &= a_1 x' + b_1 y' + c_1 \\ y &= a_2 x' + b_2 y' + c_2 \end{aligned}\right\} \qquad (1)$$

and its extensions.

Let y_r and y'_r denote respectively $\frac{1}{r!}\frac{d^r y}{dx^r}$ and $\frac{1}{r!}\frac{d^r y'}{dx'^r}$ for all positive integral values of r. Corresponding finite increments ξ, η of x and y are connected by the relation

$$\eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots, \tag{2}$$

and corresponding increments ξ', η' of x' and y' by the relation

$$\eta' = y'_1\xi' + y'_2\xi'^2 + y'_3\xi'^3 + \ldots. \tag{3}$$

Moreover, if the increments ξ', η' are those of x', y' necessitated by the increments ξ, η, of x, y,

$$\left.\begin{aligned} \xi &= a_1\xi' + b_1\eta' \\ \eta &= a_2\xi' + b_2\eta' \end{aligned}\right\}, \tag{4A}$$

which we may also write

$$\left.\begin{aligned} (a_1b_2 - a_2b_1)\,\xi' &= b_2\xi - b_1\eta \\ (a_1b_2 - a_2b_1)\,\eta' &= a_1\eta - a_2\xi \end{aligned}\right\}. \tag{4B}$$

We do not need for present purposes the expressions for either of the sets $y_1, y_2, y_3, \ldots$ and $y'_1, y'_2, y'_3, \ldots$ in terms of the other, but it is important to notice that the expressions in question do not involve explicitly x', y' or x, y, but that

$$y_1 = \frac{a_2 + b_2 y'_1}{a_1 + b_1 y'_1},$$

$$y_2 = \frac{(a_1b_2 - a_2b_1)\,y'_2}{(a_1 + b_1 y'_1)^3},$$

and generally that we pass from y_r to y_{r+1} by a total differentiation with regard to x', *i.e.*, by operation with

$$2y'_2\frac{\partial}{\partial y'_1} + 3y'_3\frac{\partial}{\partial y'_2} + 4y'_4\frac{\partial}{\partial y'_3} + \ldots,$$

and division by $(r+1)$ times $a_1 + b_1 y'$. Herein lies the special virtue of an extended (*erweiterte*) *linear* group that its extensions alone, apart from the original linear equations, form a group—in the present case a group of the fourth order or four-parameter group, the parameters being a_1, b_1, a_2, b_2, while the complete extended group is of the sixth order, having the six parameters a_1, b_1, c_1, a_2, b_2, c_2.

The problem before us is the following. If

$$f(x, y, y_1, y_2, y_3, \ldots) \equiv F(x', y', y'_1, y'_2, y'_3, \ldots)$$

is any equivalence which holds, whatever be the relation connecting

x, y, in virtue of the transformation (1) and its extensions, it is required to express

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial y_3}, \ldots$$

as linear functions, with constant or variable coefficients, of

$$\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}, \frac{\partial F}{\partial y_1'}, \frac{\partial F}{\partial y_2'}, \frac{\partial F}{\partial y_3'}, \ldots.$$

The coefficients will, of course, be independent of the forms of f and F.

We have seen that x', y' only enter explicitly in F, if at all, in virtue of the explicit occurrence of x, y in f, and conversely. In fact,

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y'} &= b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} \end{aligned}\right\}, \tag{5A}$$

or, otherwise written,

$$\left.\begin{aligned} (a_1 b_2 - a_2 b_1) \frac{\partial}{\partial x} &= b_2 \frac{\partial}{\partial x'} - a_2 \frac{\partial}{\partial y'} \\ (a_1 b_2 - a_2 b_1) \frac{\partial}{\partial y} &= a_1 \frac{\partial}{\partial y'} - b_1 \frac{\partial}{\partial x'} \end{aligned}\right\}, \tag{5B}$$

where, for shortness, f is omitted as the subject on which $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ act, and the equivalent F as that on which $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}$ act.

In order to transform $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \ldots$, let us regard the relations (2), (3), (4), which connect finite increments of x, y, x', y'. Eliminating η and η', we have

$$\left.\begin{aligned} \xi &= a_1 \xi' + b_1 (y_1' \xi' + y_2' \xi'^2 + y_3' \xi'^3 + \ldots) \\ y_1 \xi + y_2 \xi^2 + y_3 \xi^3 + \ldots &= a_2 \xi' + b_2 (y_1' \xi' + y_2' \xi'^2 + y_3' \xi'^3 + \ldots) \end{aligned}\right\}, \tag{6A}$$

which do not involve x, y, x', y' explicitly. These two equalities cannot be independent, but are identical in meaning. For substitution from (4B) for ξ', η' in (3) gives a relation in ξ, η, and $y_1', y_2', y_3', \ldots$, which, when these last are replaced by their expressions in terms of $y_1, y_2, y_3, \ldots$, must be identical with (2), as otherwise corresponding increments ξ, η of two variables connected by a single relation would be connected by two relations, as is not the case. Let us consider the

equivalent relations (6A) in the slightly more convenient forms

$$\left.\begin{aligned} \xi &= a_1\xi' + b_1(y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots) \\ b_2\xi - b_1(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots) &= (a_1b_2 - a_2b_1)\,\xi' \end{aligned}\right\}. \qquad (6B)$$

Here, since y is a perfectly arbitrary function of x, and ξ is also perfectly arbitrary, we may, for a given x, regard $y_1, y_2, y_3, \ldots$ and ξ as quite independent variables. We must then consider $y_1', y_2', y_3', \ldots$ and ξ' as dependent variables, all but the last of them as dependent on $y_1, y_2, y_3, \ldots$ in virtue of the extensions of (1), and ξ' as dependent on these and also on ξ in virtue of either of the equivalent relations (6B) at present before us. Now, of the independent variables $y_1, y_2, y_3, \ldots, \xi$, let y_r alone receive an infinitesimal increment δy_r, and let $\delta\xi'$ be the consequent increment of ξ'. The equalities (6B) yield us the two equivalent results

$$0 = \{a_1 + b_1(y_1' + 2y_2'\xi' + 3y_3'\xi'^2 + \ldots)\}\delta\xi' + b_1\left\{\xi'\frac{\partial y_1'}{\partial y_r} + \xi'^2\frac{\partial y_2'}{\partial y_r} + \xi'^3\frac{\partial y_3'}{\partial y_r} + \ldots\right\}\delta y_r,$$

$$-b_1\xi'^r\delta y_r = (a_1b_2 - a_2b_1)\,\delta\xi'.$$

Accordingly,

$$\xi'\frac{\partial y_1'}{\partial y_r} + \xi'^2\frac{\partial y_2'}{\partial y_r} + \xi'^3\frac{\partial y_3'}{\partial y_r} + \ldots$$

$$\equiv (a_1b_2 - a_2b_1)^{-1}\,\xi'^r\{a_1 + b_1(y_1' + 2y_2'\xi' + 3y_3'\xi'^2 + \ldots)\}. \qquad (7)$$

In this, on the right, substitute for ξ its equivalent

$$\xi = a_1\xi' + b_1(y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots) \qquad (8)$$

from (4A). We then have on the right a rational integral expression in ξ', which we can expand and arrange by powers of ξ'. We are thus given in (7) the identical equality of two expansions in powers of ξ', the coefficients on the two sides being all in no way dependent on ξ'. Corresponding coefficients on left and right must then be equal. In other words, the coefficients of successive powers of ξ' on the right are, in succession, the expressions for

$$\frac{\partial y_1'}{\partial y_r}, \quad \frac{\partial y_2'}{\partial y_r}, \quad \frac{\partial y_3'}{\partial y_r}, \quad \ldots$$

in terms of $y_1', y_2', y_3', \ldots$. It appears at once that the lowest power of ξ' which occurs on the right is ξ'^r, so that $\dfrac{\partial y_r'}{\partial y_r}$ is the first of these derivatives which does not vanish. In other words, as is otherwise obvious, y_r' is the lowest of the derivatives of y' with regard to x', whose expression in terms of $y_1, y_2, y_3, \ldots$ involves y_r.

3. But in the identity (7), when the right-hand member is, as above described, expressed in terms of ξ' and expanded, we may put for ξ'^s on the two sides any quantity or operator we please, and so for every other power of ξ'. Let us then put the corresponding $\frac{\partial}{\partial y'_s}$ for each ξ'^s ($s = 1, 2, 3, \ldots, \infty$), the subject of operation being any function $F(x', y', y'_1, y'_2, y'_3, \ldots)$. We thus get on the left

$$\frac{\partial y'_1}{\partial y_r}\frac{\partial}{\partial y'_1} + \frac{\partial y'_2}{\partial y_r}\frac{\partial}{\partial y'_2} + \frac{\partial y'_3}{\partial y_r}\frac{\partial}{\partial y'_3} + \ldots,$$

i.e.,

$$\frac{\partial}{\partial y_r},$$

the subject of operation being the function $f(x, y, y_1, y_2, y_3, \ldots)$ of unaccented letters which is equivalent to F. Consequently, to obtain the equivalent of $\frac{\partial}{\partial y_r}$ in the form

$$A'_1\frac{\partial}{\partial y'_1} + A'_2\frac{\partial}{\partial y'_2} + A'_3\frac{\partial}{\partial y'_3} + \ldots,$$

which is its proper one when the subject of operation is expressed in terms of $x', y', y'_1, y'_2, y'_3, \ldots$ instead of in terms of $x, y, y_1, y_2, y_3, \ldots$, we have the following rule :—Expand

$$(a_1b_2 - a_2b_1)^{-1}\{a_1\xi' + b_1(y'_1\xi' + y'_2\xi'^2 + y'_3\xi'^3 + \ldots)\}^r \times \{a_1 + b_1(y'_1 + 2y'_2\xi' + 3y'_3\xi'^2 + \ldots)\}$$

in ascending powers of ξ', writing the power of ξ' last in each term, and then put in the expanded result, for every power ξ'^s which occurs, the corresponding $\frac{\partial}{\partial y'_s}$.

Referring to (6A), this product to be expanded is seen to be briefly expressible as

$$(a_1b_2 - a_2b_1)^{-1}\xi^r\left\{\frac{\partial\xi}{\partial\xi'} + \frac{\partial\xi}{\partial\eta'}\frac{\partial\eta'}{\partial\xi'}\right\},$$

or, more shortly still, as

$$(a_1b_2 - a_2b_1)\,\xi^{-1}\frac{d\xi}{d\xi'},$$

where ξ is supposed to be expressed, as if an actual finite increment

of x, in terms of ξ' and the derivatives y'_s, which latter are not functions of ξ'.

Now, just as we have found it convenient to take ξ'^s as a symbolic representative of $\dfrac{\partial}{\partial y'_s}$ for every positive integral s, so it occurs to take ξ^r as a symbolic representative of $\dfrac{\partial}{\partial y_r}$. Our result may then be stated as follows:—If

$$f(x, y, y_1, y_2, y_3, \ldots) \equiv F(x', y', y'_1, y'_2, y'_3, \ldots)$$

be any identity consequent on the general linear transformation (1) and its extensions, then

$$\frac{\partial}{\partial y_r} f(x, y, y_1, y_2, y_3, \ldots) \quad (r = 1, 2, 3, \ldots, \infty),$$

whose symbolical form is $\xi^r f$,

has for its equivalent

$$(a_1 b_2 - a_2 b_1)^{-1} \xi^r \frac{d\xi}{d\xi'} F(x', y', y'_1, y'_2, y'_3, \ldots),$$

where $a_1 \xi' + b_1 (y'_1 \xi' + y'_2 \xi'^2 + y'_3 \xi'^3 + \ldots)$

has to be substituted for ξ, as it would have were ξ, ξ' corresponding increments of x, x' instead of mere symbols, after which the result obtained has to be expanded in powers of ξ', and then in the expansion every power ξ'^s to be replaced by the corresponding differential operator $\dfrac{\partial}{\partial y'_s}$.

The rule for transforming any linear operator

$$P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + A_1\frac{\partial}{\partial y_1} + A_2\frac{\partial}{\partial y_2} + A_3\frac{\partial}{\partial y_3} + \ldots$$

is immediately deduced. As to the first two terms they become at once

$$(a_1 b_2 - a_2 b_1)^{-1} \left\{ (Pb_2 - Qb_1)\frac{\partial}{\partial x'} + (Qa_1 - Pa_2)\frac{\partial}{\partial y'} \right\},$$

by (5B). As to the rest of the operator, write it symbolically

$$A_1 \xi + A_2 \xi^2 + A_3 \xi^3 + \ldots,$$

multiply it by $(a_1 b_2 - a_2 b_1)^{-1} \dfrac{d\xi}{d\xi'}$,

then substitute for ξ its expansion in terms of ξ' as above, expand the result and arrange it as a series in powers of ξ', writing each such power of ξ' as the last factor in its term, and, finally, for every power ξ'^s in the expansion, write the corresponding $\frac{\partial}{\partial y'_s}$.

If P, Q, A_1, A_2, A_3, ... are constants, the transformation is thus completed. If they are functions of x, y, y_1, y_2, y_3, ..., they need for useful purposes to be expressed in terms of x', y', y'_1, y'_2, y'_3, ... by means of (1) and its extensions. But, as we shall see, this expression is automatically effected in a wide and important class of cases.

Since
$$\frac{d\xi}{d\xi'}\,\frac{d\xi'}{d\xi} = 1,$$

and since the determinant of the coefficients of ξ, η in the expressions for ξ', η' is the reciprocal of the determinant of the coefficients of ξ', η' in the expressions for ξ, η, we at once see that, as should be the case, the rule obtained for the expression of an unaccented operator as an accented one affords the exactly corresponding rule for the expression of an accented operator as unaccented.

4. Let us apply these conclusions to the transformation of what I have called in the *Phil. Trans.* (*loc. cit.* in § 1) MacMahon operators. Such operators (of four elements) are those included in the definition

$$\{\mu, \nu\,;\, m, n\}_v = \frac{1}{m}\,\Sigma\left\{(\mu+\nu s)\,Y_s^{(m)}\frac{\partial}{\partial y_{n+s}}\right\},$$

the summation being with regard to s, which assumes in turn all positive integral values not less than the greater of m and $-n+1$, m and n being integral or zero, and $Y_s^{(m)}$ denoting the coefficient of ξ^s in the expansion of

$$(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m.$$

This operator is the result of replacing in the $(\mu, \nu\,;\, m, n)$ of Major MacMahon's remarkable first paper on multilinear operators* by

$$a, b, c, d, \ldots \quad \text{by} \quad 0, y_1, y_2, y_3, \ldots,$$

* "The Theory of a Multilinear Partial Differential Operator, with Applications to the Theories of Invariants and Reciprocants" (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 61–88).

or, again, that of replacing in $(\mu+\nu m, \nu;\ m, n+m-1)$,

$$a, b, c, d, \ldots \quad \text{by} \quad y_1, y_2, y_3, y_4, \ldots .$$

We shall confine attention to cases in which m is not negative, and $m+n$ not less than unity. The summation for s in $\{\mu, \nu;\ m, n\}_y$ is then $\sum_{s=m}^{s=\infty}$, *i.e.*, no coefficient $Y_s^{(m)}$ which actually occurs in the expansion of the multinomial is absent from the summation. The elaboration of results for the excepted cases of $m+n<1$ could be added, much as in my paper to which reference has been made, but would unduly lengthen the present communication. The case $m=0$ has some speciality, and will be only partially included.

5. Symbolically expressed, as in § 3, it is clear that

$$\{\mu, \nu;\ m, n\}_y = \frac{\mu}{m}\xi^n\eta^m + \nu\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi},$$

where η stands for $y_1\xi+y_2\xi^2+y_3\xi^3+\ldots$; and that in particular

$$\{1, 0;\ m, n\}_y = \frac{1}{m}\xi^n\eta^m,$$

and

$$\{0, 1;\ m, n\}_y = \xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}.$$

We can at once then apply the rule arrived at in § 3; and obtain that the results of transforming these two last operators by the extended linear scheme (1) are respectively in symbolic form

$$\frac{1}{m}(a_1b_2-a_2b_1)^{-1}\xi^n\eta^m\frac{d\xi}{d\xi},$$

i.e.,

$$\frac{1}{m}(a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m\left(a_1+b_1\frac{d\eta'}{d\xi'}\right),$$

and

$$(a_1b_2-a_2b_1)^{-1}\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}\frac{d\xi}{d\xi'},$$

i.e.,

$$(a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^{n+1}(a_2\xi'+b_2\eta')^{m-1}\left(a_2+b_2\frac{d\eta'}{d\xi'}\right).$$

Each of these expressions expanded is a sum of multiples of terms of the two types $\xi'^p\eta'^{m+n-p}$, $\xi'^p\eta'^{m+n-p}\frac{d\eta'}{d\xi'}$, *i.e.*, a sum of multiples of MacMahon y'-operators with $m'+n'=m+n$. In fact, if, as usual, $\binom{n}{r}$ denote the number of combinations of n things r together, the

results of the transformation are, when n as well as m is not negative,

$$m\,(a_1b_2-a_2b_1)\,\{1,\,0\,;\;m,\,n\}_y$$

$$\equiv \sum_{r=0,\,s=0}^{r=n,\,s=m}\binom{n}{r}\binom{m}{s}a_1^{r+1}b_1^{n-r}a_2^{s}b_2^{m-s}\xi'^{r+s}\eta'^{m+n-r-s}$$

$$+\sum_{r=0,\,s=0}^{r=n,\,s=m}\binom{n}{r}\binom{m}{s}a^{r}b^{n-r+1}a_2^{s}b_2^{m-s}\xi'^{r+s}\eta'^{m+n-r-s}\frac{d\eta'}{d\xi'}$$

$$\equiv \sum_{r=0,\,s=0}^{r=n,\,s=m}\binom{n}{r}\binom{m}{s}a_1^{r+1}b_1^{n-r}a_2^{s}b_2^{m-s}\,(m+n-r-s)$$
$$\times\{1,\,0;\;m+n-r-s,r+s\}_{y'}$$

$$+\sum_{r=0,\,s=0}^{r=n,\,s=m}\binom{n}{r}\binom{m}{s}a_1^{r}b_1^{n-r+1}a_2^{s}b_2^{m-s}\,\{0,\,1\,;\;m+n-r-s+1,\,r+s-1\}_{y'},$$

and
$$(a_1b_2-a_2b_1)\,\{0,\,1\,;\;m,\,n\}_y$$

$$\equiv \sum_{r=0,\,s=0}^{r=n+1,\,s=m-1}\binom{n+1}{r}\binom{m-1}{s}a_1^{r}b_1^{n+1-r}a_2^{s+1}b_2^{m-1-s}\,(m+n-r-s)$$
$$\times\{1,\,0\,;\;m+n-r-s,\,r+s\}_{y'}$$

$$+\sum_{r=0,\,s=0}^{r=n+1,\,s=m-1}\binom{n+1}{r}\binom{m-1}{s}a_1^{r}b_1^{n+1-r}a_2^{s}b_2^{m-s}$$
$$\times\{0,\,1\,;\;m+n-r-s+1,\,r+s-1\}_{y'}.$$

The transformation of $\{\mu,\,\nu\,;\;m,\,n\}_y$ is immediately deduced by adding $\frac{\mu}{m}$ times the former of these results to ν times the latter.

6. Such formulæ have necessarily some cumbrousness of expression; and clearness of realization is, I am sure, gained by thinking of them as before us in their symbolical forms, namely,

$$\xi^n\eta^m \equiv (a_1b_2-a_2b_1)^{-1}\,\xi^n\eta^m\frac{d\xi}{d\xi'}$$
$$=(a_1b_2-a_2b_1)^{-1}\,(a_1\xi'+b_1\eta')^n\,(a_2\xi'+b_2\eta')^m\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)$$

and

$$\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi} \equiv (a_1b_2-a_2b_1)^{-1}\,\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi'}$$
$$=(a_1b_2-a_2b_1)^{-1}\,(a_1\xi'+b_1\eta')^{n+1}\,(a_2\xi'+b_2\eta')^{m-1}\left(a_2+b_2\frac{d\eta'}{d\xi'}\right),$$

where before interpretation expansion on the left has to be in powers of ξ, and on the right in powers of ξ'.

The interpretation presents no difficulty when m as well as $m+n$

is a positive integer. Moreover, the value $m = 0$ has no speciality as far as the transformation of $\xi^n\eta^m$ is concerned; but this value has in connexion with the transformation of $\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}$. As it is not proposed to deal with this, the second of the two above transformations is better written

$$\xi^n\eta^m\frac{d\eta}{d\xi} \equiv (a_1b_2-a_2b_1)^{-1}\xi^n\eta^m\frac{d\eta}{d\xi'}$$
$$= (a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m\left(a_2+b_2\frac{d\eta'}{d\xi'}\right),$$

in which, as in the first, m is a positive integer or zero, and $m+n$ a positive integer. We will, in fact, only attend to cases in which neither m nor n is negative in the last article, the cases to which the forms of equalities written at the end of that article apply.

7. Let us speak of an operator which is a linear function with constant coefficients of operators of the two types $\xi^n\eta^m$, $\xi^n\eta^m\frac{d\eta}{d\xi}$ with the same value of $m+n$ as being of the $(m+n)^{\text{th}}$ order. The results before us involve the fact that it transforms into another operator of the same order $m+n$.

Some facts with regard to the linearly independent operators

$$\xi,\ \eta,\ \xi\frac{d\eta}{d\xi},\ \eta\frac{d\eta}{d\xi},\quad i.e.,\ \tfrac{1}{2}\frac{d}{d\xi}(\eta^2),$$

of the first order will now be adduced. Written at length they are respectively

$$\xi = \frac{\partial}{\partial y_1},$$
$$\eta = y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}+\ldots,$$
$$\xi\frac{d\eta}{d\xi} = y_1\frac{\partial}{\partial y_1}+2y_2\frac{\partial}{\partial y_2}+3y_3\frac{\partial}{\partial y_3}+\ldots,$$
$$\eta\frac{d\eta}{d\xi} = 2.\tfrac{1}{2}y_1^2\frac{\partial}{\partial y_1}+3y_1y_2\frac{\partial}{\partial y_2}+4\,(y_1y_3+\tfrac{1}{2}y_2^2)\frac{\partial}{\partial y_3}+5\,(y_1y_4+y_2y_3)\frac{\partial}{\partial y_4}$$
$$+6\,(y_1y_5+y_2y_4+\tfrac{1}{2}y_3^2)\frac{\partial}{\partial y_5}+\ldots$$
$$= -y_1^2\xi+y_1\left(\eta+\xi\frac{d\eta}{d\xi}\right)+V,$$

where V is Sylvester's operator

$$4.\tfrac{1}{2}y_2^2\frac{\partial}{\partial y_3}+5y_2y_3\frac{\partial}{\partial y_4}+6\,(y_2y_4+\tfrac{1}{2}y_3^2)\frac{\partial}{\partial y_5}+7\,(y_2y_5+y_3y_4)\frac{\partial}{\partial y_6}+\dots .$$

They are recognized as being the known operators which determine the four independent infinitesimal transformations of the group which consists of the extensions of the general linear group (1). (A more general fact including this will be proved in § 16 below.) The general infinitesimal transformation of this group of extensions is determined by

$$\lambda\xi+\mu\eta+\nu\xi\frac{d\eta}{d\xi}+\varpi\eta\frac{d\eta}{d\xi}.$$

i.e., by the general operator of the first order. The functions of the derivatives $y_1, y_2, y_3, \dots$ which are absolute differential invariants of the group of extensions are exactly those functions which have the four operators for annihilators. They are also the absolute differential invariants of the extended linear group (1) itself, as the two annihilators of such invariants given by infinitesimal variation of c_1 and c_2 in the first place are $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$, which show that there is no differential invariant of the group which involves x or y.

Now, a function $f(y_1, y_2, y_3, \dots)$ annihilated by ξ or $\dfrac{\partial}{\partial y_1}$ is free from y_1; one annihilated by η is homogeneous of degree zero; one annihilated by $\xi\dfrac{d\eta}{d\xi}$ is isobaric of weight zero, weight being measured by sum of suffixes; and one annihilated by $\eta\dfrac{d\eta}{d\xi}$ in addition to the above, or, less exactingly, in addition to ξ and $\eta+\xi\dfrac{d\eta}{d\xi}$, is annihilated by V.

Consider now the transformations of these operators of the first order, as given by § 6. They are

$$\xi\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_1^2\xi'+a_1b_1\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+b_1^2\eta'\frac{d\eta'}{d\xi'}\right\},$$

$$\eta\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_1a_2\xi'+a_1b_2\eta'+a_2b_1\xi'\frac{d\eta'}{d\xi'}+b_1b_2\eta'\frac{d\eta'}{d\xi'}\right\},$$

$$\xi\frac{d\eta}{d\xi}\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_1a_2\xi'+a_2b_1\eta'+a_1b_2\xi'\frac{d\eta'}{d\xi'}+b_1b_2\eta'\frac{d\eta'}{d\xi'}\right\},$$

$$\eta\frac{d\eta}{d\xi}\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_2^2\xi'+a_2b_2\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+b_2^2\eta'\frac{d\eta'}{d\xi'}\right\}.$$

We notice here a difference of character between the middle pair and the extreme pair—the coefficients of ξ', η', $\xi'\frac{d\eta'}{d\xi'}$, $\eta'\frac{d\eta'}{d\xi'}$ are linearly independent in the one pair of cases and not so in the other pair. Look, as we of course may do, upon the equivalences as results of applying a general linear transformation to the accented variables instead of to the unaccented, *i.e.*, look upon $f(y_1, y_2, y_3, \ldots)$ as having been found as the equivalent of a function $F(y_1', y_2', y_3', \ldots)$ and not *vice versa*. We see that for η to annihilate f, whatever the linear transformation may have been, or for $\xi\frac{d\eta}{d\xi}$ to annihilate it in like case, it is necessary for all four of ξ', η', $\xi'\frac{d\eta'}{d\xi'}$, $\eta'\frac{d\eta'}{d\xi'}$ to annihilate F. On the other hand, for ξ or for $\eta\frac{d\eta}{d\xi}$ to annihilate f, whatever the linear transformation may have been, it is only necessary for ξ', $\eta'+\xi'\frac{d\eta'}{d\xi'}$, $\eta'\frac{d\eta'}{d\xi'}$ to annihilate F.

We may therefore state as follows. In order that a function of the derivatives may become a homogeneous function of degree zero of the new derivatives after every linear transformation of the variables, or in order that it may become an isobaric function of weight zero, it is necessary that it have the four properties of being free from the first derivative y_1, homogeneous of degree zero, isobaric of weight zero, and annihilated by $\eta\frac{d\eta}{d\xi}$ or V; in other words, it must be an absolute differential invariant of the general linear group. But, in order that it may become, after every linear transformation of the variables, a function free from the first derivative, or a function annihilated by V, it is only necessary for it to have the three properties of being free from y_1, being of zero sum of degree and weight throughout, and being annihilated by V; in other words, it need be only an absolute pure reciprocant (or linear function of such), *i.e.*, an absolute differential invariant (or linear function of such) of a linear group whose generality is limited by the one relation $a_1b_2-a_2b_1$ among the coefficients—a sub-group of the general linear group.*

* By "absolute differential invariant" of a continuous group, I, in this paper, mean, with Lie, a function of derivatives, and it might be also of the variables—though these last do not occur in the cases of groups here considered—which is absolutely unaltered in form by the substitutions of the group and its extensions. A function which persists in form but for a factor is called a "relative or non-absolute differential invariant." By "absolute pure reciprocant," I mean, with

8. It is an interesting subject of investigation whether there are MacMahon operators which transform, by the general linear scheme, into themselves, as there are for mere interchange of the variables and some other very special schemes. Were there such, and were the coefficients in them free from the constants in the scheme, then any such operator would generate absolute differential invariants from absolute differential invariants. But the answer appears, for

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Sylvester, a function of second and higher derivatives which persists in form but for the assumption of a factor which is a power of -1 when dependent and independent variables are interchanged. A non-absolute pure reciprocant persists in form after such interchange except for a factor which is not a mere power of -1.

It is a prevalent, but mistaken, impression that absolute pure reciprocants are identical with absolute differential invariants of the general linear group. What Sylvester proved (*Amer. Jour.*, Vol. VIII., pp. 248, &c.) with regard to the differential invariancy of a pure reciprocant R for the general linear substitution is that, if homogeneous (of degree i), and consequently isobaric (of weight w), it becomes $(a_1b_2-a_2b_1)^i(a_1+b_1y_1)^{-w-i}R$, with my notation as above: his w is different. If it be an *absolute* reciprocant, what he has proved at an earlier stage is that $w+i=0$, but not that $i=0$, $w=0$ separately. There is absolute invariancy for the substitution only if $i=0$ as well as $w+i=0$. There are, however, absolute pure reciprocants of all degrees i, and even non-homogeneous ones. Sylvester calls those of degree zero *plenarily absolute* to indicate that they are absolutely invariant for any linear group. Absolute pure reciprocants which are not of degree zero are, if homogeneous, differential invariants of any linear group, but are absolute differential invariants only for linear groups in which generality of constants is limited by the one relation $a_1b_2-a_2b_1=1$.

Again, what I myself proved (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 388) with regard to the differential invariancy of pure cyclicants or ternary reciprocants for the general linear substitution of $lx+my+nz+p$, &c., for x, &c., is that there is persistence in the case of a homogeneous one but for a factor of the form

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}^i \left\{lm'-l'm-(mn'-m'n)\frac{\partial z}{\partial x}-(nl'-n'l)\frac{\partial z}{\partial y}\right\}^{-i-\frac{1}{3}w}.$$

The second index here vanishes when the pure cyclicant is absolute; but the first i, as a rule, does not. Thus absolute pure cyclicants include, but are not only coextensive with, all absolute differential invariants of the general linear group in three variables. Linear functions of them are identical with absolute differential invariants of a sub-group of the general linear group, namely, of the *special* sub-group in which the generality of coefficients is limited by the one relation

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 1.$$

Those who, like myself, in 1886 and following years, followed Sylvester in researches on reciprocants and allied classes of differential invariants, and showed, it must be confessed, imperfect acquaintance with Sophus Lie's grand work on Continuous Groups and Differential Invariants in general, have reason to be grateful to Herr O. A. Stöckert for a recent memoir, *Ueber die Beziehungen der Reciprokantentheorie zur allgemeinen Theorie der Differentialinvarianten* (Chemnitz, 1895), in which he elucidates very instructively Lie's theory in its bearing on Sylvester's and our own. In that memoir the error occurs (pp. 30, 32) of attributing to Sylvester the opinion that an absolute pure reciprocant is necessarily of zero degree and weight. The proof given fails on p. 32 at line 11.

practical purposes, negative. It is true that inspection of the symbolic equivalences of the last article gives us one such persistent operator of the first order

$$\xi\frac{d\eta}{d\xi}-\eta\equiv\xi'\frac{d\eta'}{d\xi'}-\eta',$$

i.e., $$y_2\frac{\partial}{\partial y_2}+2y_3\frac{\partial}{\partial y_3}+3y_4\frac{\partial}{\partial y_4}+\ldots\equiv y_2'\frac{\partial}{\partial y_2'}+2y_3'\frac{\partial}{\partial y_3'}+3y_4'\frac{\partial}{\partial y_4'}+\ldots,$$

so that the persistence (19) of my memoir in the *Phil. Trans.* (*loc. cit.* p. 25) is one which holds in general; but this has merely the effect of multiplying a homogeneous isobaric function by the excess of its weight over its degree. It expresses the persistence in value of the characteristic $w-i$, but gives, in cases of interest, nothing new by its operation (a conclusion drawn by means of it which, though of interest, is not new, is the case for $q=2$ of one in § 17 below). The more general operator $\xi^{m-1}\eta^{m-1}\left\{\xi\frac{d\eta}{d\xi}-\eta\right\}$ [*loc. cit.*, (17)], which is persistent in the special theory, is not so in the present general one except for $m=n=1$.

Nor do the operators $\xi\frac{d\eta}{d\xi}+\eta$, $\xi^{m-1}\eta^{m-1}\left\{\xi\frac{d\eta}{d\xi}+\eta\right\}$, which are negatively persistent in the special theory [*loc. cit.*, (20), (18)], yield directly anything of much interest in general. If we take the general operator of the first order,

$$\lambda\xi+\mu\eta+\nu\xi\frac{d\eta}{d\xi}+\varpi\eta\frac{d\eta}{d\xi},$$

and try to find all values of λ, μ, ν, ϖ for which the form persists but for a factor, upon substitution from the last article, we obtain, besides the persistent $\xi\frac{d\eta}{d\xi}-\eta$ as above, only the equivalences

$$-2a_2\xi+(a_1-b_2)\left(\eta+\xi\frac{d\eta}{d\xi}\right)+2b_1\eta\frac{d\eta}{d\xi}$$
$$\equiv-2a_2\xi'+(a_1-b_2)\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+2b_1\eta'\frac{d\eta'}{d\xi'}$$

and $$(a_1-b_2)\left(a_2\xi-b_1\eta\frac{d\eta}{d\xi}\right)+2a_2b_1\left(\eta+\xi\frac{d\eta}{d\xi}\right)$$
$$\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\left(a_2\xi+b_1\eta\frac{d\eta}{d\xi}\right)$$
$$\equiv\frac{\left\{a_1+b_2\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\right\}^2}{4(a_1b_2-a_2b_1)}\left\{(a_1-b_2)\left(a_2\xi'-b_1\eta'\frac{d\eta'}{d\xi'}\right)\right.$$
$$\left.+2a_2b_1\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\left(a_2\xi'+b_1\eta'\frac{d\eta'}{d\xi'}\right)\right\},$$

which, involving as they do a_1, b_1, a_2, b_2 as coefficients, give us no information as to persistence for all linear transformations, so that they are only matters of curiosity in general. It is the latter, however, which, for the case of $a_1 = b_2 = 0$, $a_2 = b_1 = 1$, *i.e.*, for the case of mere interchange of variables x, y, gives the negative persistence of $\eta + \xi \frac{d\eta}{d\xi}$, *i.e.*,

$$2y \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots$$

[*loc. cit.*, (20)]. It also gives for that case the negative persistence of $\xi + \eta \frac{d\eta}{d\xi}$, *i.e.*, of

$$(1-y_1^2) \frac{\partial}{\partial y_1} + y_1 \left(2y_1 \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots \right) + V,$$

and the former gives the positive persistence of $\eta \frac{d\eta}{d\xi} - \xi$, *i.e.*, of

$$-(1+y_1^2) \frac{\partial}{\partial y_1} + y_1 \left(2y_1 \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots \right) + V,$$

two facts not expressly introduced in the memoir to which reference is made.

9. The digression may be pardonable if I consider for a moment some properties of operators of the types $\xi^n \eta^m$, $\xi^n \eta^m \frac{d\eta}{d\xi}$ without reference to transformations. To find their *alternants* will really be to do otherwise what MacMahon has already done (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 66–69), but the use of the present compact symbolic forms so simplifies the process, and exhibits the results in such suggestive shape, that I believe there to be justification.

Let us denote by $(A\xi + B\xi^2 + \dots)(A'\xi + B'\xi^2 + \dots)$ the operator which results from the performance of the operation represented by the first written factor on the back of that represented by the second. The part of the resultant operator which does not involve symbols of second partial differentiation will be obtained in symbolic form by making the left-hand operator act only on A', B', ..., dealing with the ξ in the right-hand operator as if it were a constant. Thus, in particular,

$$(A\xi + B\xi^2 + C\xi^3 + \dots)\,\eta = (A\xi + B\xi^2 + C\xi^3 + \dots)(y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots)$$
$$= A\xi + B\xi^2 + C\xi^3 + \dots + R,$$

where R involves $\dfrac{\partial^2}{\partial y_1^2}$, $\dfrac{\partial^2}{\partial y_1 \partial y_2}$, ... So

$$(A\xi+B\xi^2+C\xi^3+\ldots)\,\xi^n\eta^m = m\xi^n\eta^{m-1}(A\xi+B\xi^2+C\xi^3+\ldots)+R';$$

and $$(A\xi+B\xi^2+C\xi^3+\ldots)\,\xi^n\eta^m\frac{d\eta}{d\xi} = m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}(A\xi+B\xi^2+C\xi^3+\ldots)$$
$$+\xi^n\eta^m\frac{d}{d\xi}(A\xi+B\xi^2+C\xi^3+\ldots)+R''.$$

Now, ϑ, ϕ being any two linear operators, we call as usual $\vartheta\phi-\phi\vartheta$ the alternant of ϑ and ϕ, and write it $(\vartheta.\phi)$. The terms R in $\vartheta\phi$ and $\phi\vartheta$ are the same. Hence we at once deduce

$$\begin{aligned}(\xi^\nu\eta^\mu.\xi^n\eta^m) &= \xi^\nu\eta^\mu.\xi^n\eta^m-\xi^n\eta^m.\xi^\nu\eta^\mu\\ &= m\xi^n\eta^{m-1}\xi^\nu\eta^\mu-\mu\xi^\nu\eta^{\mu-1}\xi^n\eta^m\\ &= (m-\mu)\,\xi^{n+\nu}\eta^{m+\mu-1},\end{aligned}$$

$$\left(\xi^\nu\eta^\mu.\xi^n\eta^m\frac{d\eta}{d\xi}\right) = m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}\xi^\nu\eta^\mu+\xi^n\eta^m\frac{d}{d\xi}(\xi^\nu\eta^\mu)-\mu\xi^\nu\eta^{\mu-1}\xi^n\eta^m\frac{d\eta}{d\xi}$$
$$= m\xi^{n+\nu}\eta^{m+\mu-1}\frac{d\eta}{d\xi}+\nu\xi^{n+\nu-1}\eta^{m+\mu},$$

$$\left(\xi^\nu\eta^\mu\frac{d\eta}{d\xi}.\xi^n\eta^m\frac{d\eta}{d\xi}\right) = m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}\xi^\nu\eta^\mu\frac{d\eta}{d\xi}+\xi^n\eta^m\frac{d}{d\xi}\left(\xi^\nu\eta^\mu\frac{d\eta}{d\xi}\right)$$
$$-\mu\xi^\nu\eta^{\mu-1}\frac{d\eta}{d\xi}\xi^n\eta^m\frac{d\eta}{d\xi}-\xi^\nu\eta^\mu\frac{d}{d\xi}\left(\xi^n\eta^m\frac{d\eta}{d\xi}\right)$$
$$= (\nu-n)\,\xi^{n+\nu-1}\eta^{m+\mu}\frac{d\eta}{d\xi}.$$

Thus the alternant of two MacMahon operators of given orders $(m+n,\ \mu+\nu)$ is a MacMahon operator whose order $(m+n+\mu+\nu-1)$ is one less than the sum of the orders of the two.

Moreover, the alternant of two operators of the same type $\left(\xi^\nu\eta^\mu \text{ or } \xi^n\eta^m\dfrac{d\eta}{d\xi}\right)$ is an operator of the same type. The alternant of two of different types $\left(\xi^\nu\eta^\mu \text{ and } \xi^n\eta^m\dfrac{d\eta}{d\xi}\right)$ is as a rule the sum of two operators, one of each type.

It is easy to see, by consideration of the indices and coefficients in the three types of alternant equalities that η is the only operator of positive integral order of the type $\xi^n\eta^m$ which cannot be expressed as

an alternant of operators of the same type, and that $\xi\frac{d\eta}{d\xi}$ is the only one of the type $\xi^r\eta^m\frac{d\eta}{d\xi}$ which cannot be expressed as an alternant of operators of its type. It is also easy to see that the alternant $\left(\xi.\eta\frac{d\eta}{d\xi}\right)$ is the only one of the type $\left(\xi^r\eta^s.\xi^n\eta^m\frac{d\eta}{d\xi}\right)$ which cannot be linearly expressed in terms of alternants of the two types $(\xi^r\eta^s.\xi^n\eta^m)$, $\left(\xi^r\eta^s\frac{d\eta}{d\xi}.\xi^n\eta^m\frac{d\eta}{d\xi}\right)$.

The alternants of operators of the first order are themselves of the first order, *i.e.*, are linear functions of themselves, a fact which expresses that, as mentioned earlier, they possess the group property. They are

$$(\xi.\eta)=\xi,$$

$$\left(\xi.\xi\frac{d\eta}{d\xi}\right)=\xi,$$

$$\left(\xi.\eta\frac{d\eta}{d\xi}\right)=\eta+\xi\frac{d\eta}{d\xi},$$

$$\left(\eta.\xi\frac{d\eta}{d\xi}\right)=0,$$

$$\left(\eta.\eta\frac{d\eta}{d\xi}\right)=\eta\frac{d\eta}{d\xi},$$

$$\left(\xi\frac{d\eta}{d\xi}.\eta\frac{d\eta}{d\xi}\right)=\eta\frac{d\eta}{d\xi}.$$

It is, moreover, evident hence that, as also stated earlier, ξ, $\eta+\xi\frac{d\eta}{d\xi}$, $\eta\frac{d\eta}{d\xi}$ form a sub-group, as their alternants in pairs are linear in themselves.

What the group and sub-group are has been already indicated (§ 7).

10. To return to transformations by the extended linear scheme (1). The transformation of MacMahon operators of positive integral order is what we have so far considered in application of the method of § 3. But the method is one which, as we saw, applies to all linear operators whatever. One simple MacMahon operator of zero order may be mentioned in passing, namely,

$$\frac{d\eta}{d\xi}-y_1\equiv 2y_2\frac{\partial}{\partial y_1}+3y_3\frac{\partial}{\partial y_2}+4y_4\frac{\partial}{\partial y_3}+\ldots,$$

whose effect is that of $\frac{d}{dx}$, the symbol of total differentiation with regard to x in so far as $y_1, y_2, y_3, \ldots$ are functions of x in virtue of the relation connecting x and y. After substitution for y_1 in terms of y_1', by § 1, its transformation becomes by the general rule

$$\frac{1}{a_1+b_1y_1'}\left\{\frac{d\eta'}{d\xi'}-y_1'\right\},$$

which accords with the facts of total differentiation.

We may transform linear operators whose symbolical expressions involve $\frac{d^2\eta}{d\xi^2}$, $\frac{d^3\eta}{d\xi^3}$, ..., in so far as we know the expressions for those derivatives in terms of $\frac{d\eta'}{d\xi'}$, $\frac{d^2\eta'}{d\xi'^2}$, $\frac{d^3\eta'}{d\xi'^3}$, ..., when ξ, η, ξ', η' are actual increments of x, y, x', y'. The expressions in question are of the same form as those (§ 2) for $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... in terms of $\frac{dy'}{dx'}$, $\frac{d^2y'}{dx'^2}$, $\frac{d^3y'}{dx'^3}$, ... Thus, for instance, since

$$\frac{d^2\eta}{d\xi^2}\equiv(a_1b_2-a_2b_1)\frac{d^2\eta'}{d\xi'^2}\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{-3}\equiv(a_1b_2-a_2b_1)\frac{d^2\eta'}{d\xi'^2}\left(\frac{d\xi}{d\xi'}\right)^{-3},$$

we have, by § 3, that the transformed equivalent of

$$\xi^n\frac{d^2\eta}{d\xi^2}=1.2y_2\frac{\partial}{\partial y_n}+2.3y_3\frac{\partial}{\partial y_{n+1}}+3.4y_4\frac{\partial}{\partial y_{n+2}}+\ldots\quad(n>0)$$

is, symbolically, the expansion in powers of ξ' of

$$\xi^n\frac{d^2\eta'}{d\xi'^2}\left(\frac{d\xi}{d\xi'}\right)^{-2}=\left\{a_1\xi'+b_1(y_1'\xi'+y_2'\xi'^2+\ldots)\right\}^n$$

$$\times\left\{1.2y_2'+2.3y_3'\xi'+3.4y_4'\xi'^2+\ldots\right\}\left\{a_1+b_1(y_1'+2y_2'\xi'+3y_3'\xi'^2+\ldots)\right\}^{-2},$$

and, actually, the result of replacing, in the expansion,

$$\xi'^r \text{ by } \frac{\partial}{\partial y_r'}\quad(r=1, 2, 3, \ldots, \infty);$$

and more generally that the transformed equivalent of

$$\xi^n\eta^m\left(\frac{d\eta}{d\xi}\right)^p\left(\frac{d^2\eta}{d\xi^2}\right)^q$$

is, symbolically,

$$(a_1b_2-a_2b_1)^{q-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m$$

$$\times\left(a_2+b_2\frac{d\eta'}{d\xi'}\right)^p\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{1-p-3q}\left(\frac{d^2\eta'}{d\xi'^2}\right)^q.$$

An interesting class of examples of such transformations is afforded by our knowledge of pure reciprocants. If $R\left(\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \frac{d^4\eta}{d\xi^4}, \ldots\right)$ be a pure reciprocant in ξ, η of degree i and weight (sum of indices of differentiation in each term) w, we know (§ 7, footnote) that its expression in terms of ξ', η' (regarded as actual and not symbolic) is

$$(a_1 b_2 - a_2 b_1)^i \left(a_1 + b_1 \frac{d\eta'}{d\xi'}\right)^{-w-i} R\left(\frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \frac{d^4\eta'}{d\xi'^4}, \ldots\right).$$

Hence the transformed equivalent of the operator whose symbolic expression is the expansion of

$$\xi^n R\left(\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \frac{d^4\eta}{d\xi^4}, \ldots\right) \quad (n>0),$$

where η denotes $y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots$, is, symbolically, the expansion of

$$(a_1 b_2 - a_2 b_1)^{i-1} (a_1\xi' + b_1\eta')^n \left(a_1 + b_1 \frac{d\eta'}{d\xi'}\right)^{1-w-i} R\left(\frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \frac{d^4\eta'}{d\xi'^4}, \ldots\right),$$

where η' denotes $y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots$.

II.

11. We now proceed to the more general analogous theory when there are q variables supposed connected by one relation. Let $q-1$ of them, chosen as the independent variables, be called $x_1, x_2, \ldots, x_{q-1}$, and the single dependent one y. After the preceding exposition of the first case of $q = 2$, a much slighter general presentation will suffice than would have been necessary had a different order been chosen, and the inevitable imperfections of a notation for partial differential coefficients, and the necessary avoidance of extensively writing out explicit forms of operators dealt with, will, it is hoped, cause but little obscurity.

The notation $y_{rst\ldots}$ will be used as denoting

$$\frac{1}{r!\,s!\,t!\ldots r_{q-1}!} \frac{\partial^{r+s+t+\ldots+r_{q-1}}}{\partial x_1^r \partial x_2^s \partial x_3^t \ldots \partial x_{q-1}^{r_{q-1}}} y.$$

There are supposed to be always $q-1$ suffixes, each of which may be zero or any positive integer, while their sum is 1 at least.

Let $\xi_1, \xi_2, \xi_3, \ldots, \xi_{q-1}$ be independent simultaneous finite increments of the $q-1$ independent variables, and η the corresponding increment of y. It is supposed that Taylor's theorem applies to y as a function

of $x_1, x_2, \ldots, x_{q-1}$ for values considered, so that

$$\eta = \sum_{r+s+t+\ldots+r_{q-1}=1}^{r=\infty,\, s=\infty,\, t=\infty,\, \ldots,\, r_{q-1}=\infty} y_{rst\ldots}\, \xi_1^r \xi_2^s \xi_3^t \ldots \xi_{q-1}^{r_{q-1}}.$$

As well as using $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in an actual sense, we shall also use them purely as symbols with a meaning now to be explained. The symbol η will always mean the above expansion in terms of the symbols ξ. These latter symbols will have no individual meaning except when standing alone in the first power in an ultimate expansion. A power or product of symbols ξ in an ultimate expansion will have a meaning as a whole, namely,

$$\xi_1^r \xi_2^s \xi_3^t \ldots \xi_{q-1}^{r_{q-1}} \text{ will mean } \frac{\partial}{\partial y_{rst\ldots r_{q-1}}},$$

a symbol of partial differentiation, the supposition being that it is acting on a function of the various partial derivatives of y, and, it may be, of the variables $y, x_1, x_2, \ldots, x_{q-1}$ in addition. Thus, for instance,

$$\xi_1, \xi_2, \xi_3, \xi_2^2, \xi_1^2 \xi_3, \ldots$$

will mean respectively

$$\frac{\partial}{\partial y_{100\ldots}}, \frac{\partial}{\partial y_{010\ldots}}, \frac{\partial}{\partial y_{001\ldots}}, \frac{\partial}{\partial y_{020\ldots}}, \frac{\partial}{\partial y_{201\ldots}}, \ldots.$$

By an operator such as

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}}\, \eta^m \frac{\partial \eta}{\partial \xi_1},$$

for instance, we shall mean the operator obtained by substituting in the product for η its expansion in terms of the ξs as above, performing the partial differentiation with regard to ξ_1, expanding the resulting product in powers and products of powers of $\xi_1, \xi_2, \ldots, \xi_{q-1}$, writing every such power or product of powers after its multiplier in the expansion, and finally substituting for each product $\xi_1^r \xi_2^s \xi_3^t \ldots$ as it occurs ultimately the corresponding symbol of partial differentiation as above.

12. We shall have to consider largely operators of the q types

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m,\ \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_1},\ \ldots,\ \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_{q-1}},$$

where $n_1, n_2, \ldots, n_{q-1}, m$ are positive, integral, or zero, and $\Sigma n + m \neq 1$. These operators are for the case $q = 2$ the MacMahon operators of

the preceding section, and, for the next case $q = 3$ the analogous ternary operators which I have used on a previous occasion (*Phil. Trans.*, *loc. cit.*, pp. 36, &c.). We speak of an operator which is a linear function with constant coefficients of operators of these types for which $\Sigma n + m$ is constant as being of order $\Sigma n + m$.

The alternants of pairs of operators of these types are with ease written down as in § 9. There are three classes of fundamental alternant equalities: viz.,

$$(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}.\,\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m})$$

$$= (m-\mu)\,\xi_1^{n_1+\nu_1}\xi_2^{n_2+\nu_2}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu-1},$$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}.\,\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$

$$= m\xi_1^{n_1+\nu_1}\xi_2^{n_2+\nu_2}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu-1}\frac{\partial\eta}{\partial\xi_s}$$

$$+\nu_s\xi_1^{n_1+\nu_1}\dots\xi_s^{n_s+\nu_s-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu},$$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}.\,\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$

$$= m\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m-1}\frac{\partial\eta}{\partial\xi_s}\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}$$

$$+\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial}{\partial\xi_s}\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}\right)$$

$$-\mu\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu-1}\frac{\partial\eta}{\partial\xi_r}\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}$$

$$-\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial}{\partial\xi_r}\left(\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$

$$= \nu_s\xi_1^{n_1+\nu_1}\dots\xi_s^{n_s+\nu_s-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_r}$$

$$-n_r\xi_1^{n_1+\nu_1}\dots\xi_r^{n_r+\nu_r-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_s}$$

of which the last includes in particular ($r = s$)

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}.\,\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_r}\right)$$

$$= (\nu_r-n_r)\,\xi_1^{n_1+\nu_1}\dots\xi_r^{n_r+\nu_r-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_r}.$$

On inspection of these identities we at once see that the alternant of two operators of given orders $\Sigma\nu+\mu$, $\Sigma n+m$ is an operator of the same kind whose order $\Sigma\nu+\Sigma n+\mu+m-1$ is one less than the sum of those given orders.

In particular, the alternant of two operators of the first order is itself an operator of the first order. The linearly independent operators of the first order are q^2 in number: viz., $q-1$ operators ξ_r $(r=1, 2, \ldots, q-1)$, one operator η, $q-1$ operators of the type $\xi_1\frac{\partial\eta}{\partial\xi_1}$, $(q-1)(q-2)$ operators of the type $\xi_1\frac{\partial\eta}{\partial\xi_2}$, and $q-1$ operators $\eta\frac{\partial\eta}{\partial\xi_r}$ $(r=1, 2, \ldots, q-1)$. The alternant identities for them in pairs are of the following types, the number of each type being written after every one, an alternant $(\vartheta.\phi)$ not being reckoned as distinct from its negative $(\phi.\vartheta)$;

$$(\xi_1.\xi_2)=0,\ \tfrac{1}{2}(q-1)(q-2); \qquad (\xi_1.\eta)=\xi_1,\ q-1;$$

$$\left(\xi_1.\xi_1\frac{\partial\eta}{\partial\xi_1}\right)=\xi_1,\ q-1; \qquad \left(\xi_1.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=0,(q-1)(q-2);$$

$$\left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2,\ (q-1)(q-2); \qquad \left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3); \qquad \left(\xi_1.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\xi_1\frac{\partial\eta}{\partial\xi_1}+\eta,\ q-1;$$

$$\left(\xi_1.\eta\frac{\partial\eta}{\partial\xi_2}\right)=\xi_1\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2); \qquad \left(\eta.\xi_1\frac{\partial\eta}{\partial\xi_1}\right)=0,\ q-1;$$

$$\left(\eta.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2); \qquad \left(\eta.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_1},\ q-1;$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=-\xi_1\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2\frac{\partial\eta}{\partial\xi_1},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_1},\ q-1; \qquad \left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2\frac{\partial\eta}{\partial\xi_2}-\xi_1\frac{\partial\eta}{\partial\xi_1},\ \tfrac{1}{2}(q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_1\frac{\partial\eta}{\partial\xi_3}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_1}\right)=\xi_3\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_4}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3)(q-4);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3);$$

$$\left(\eta\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2).$$

The whole number of these equalities is $\frac{1}{2}q^2(q^2-1)$.

As the alternants in pairs of the q^2 linearly independent operators of the first order are linear with constant coefficients in the operators themselves, except such of them as vanish, they are the operators appertaining to the infinitesimal transformations of a continuous substitution group of q^2 parameters. It will be seen later (§ 16) that the group in question is that which consists of the extensions of the general linear group of substitutions for the q variables $x_1, x_2, \ldots, x_{q-1}, y$.

Moreover, we notice that the $q-1$ operators ξ, the $(q-1)(q-2)$ operators $\xi_r\frac{\partial\eta}{\partial\xi_s}$ $(r\neq s)$, and the $q-1$ operators $\eta\frac{\partial\eta}{\partial\xi}$ all occur as alternants of operators; but that the operator η and the $q-1$ operators $\xi_r\frac{\partial\eta}{\partial\xi_r}$ only occur in alternants in the connexions $\eta+\xi_r\frac{\partial\eta}{\partial\xi_r}$ and

$$\xi_r\frac{\partial\eta}{\partial\xi_r}-\xi_s\frac{\partial\eta}{\partial\xi_s}=\left(\eta+\xi_r\frac{\partial\eta}{\partial\xi_r}\right)-\left(\eta+\xi_s\frac{\partial\eta}{\partial\xi_s}\right).$$

The q^2-1 operators

$$\xi, \quad \xi_r \frac{\partial \eta}{\partial \xi_s} \ (r \neq s), \quad \eta \frac{\partial \eta}{\partial \xi}, \quad \eta + \xi_r \frac{\partial \eta}{\partial \xi_r} \quad (r = 1, 2, 3, \ldots, q-1)$$

are then seen to be a set whose alternants are linear with constant coefficients in themselves. These are consequently the operators appertaining to the infinitesimal transformations of a sub-group of the general group of extensions. What we see in § 16 will also show us that this sub-group is the group of extensions of the group of linear substitutions in which the generality of the coefficients of $x_1, x_2, \ldots, x_{q-1}, y$ is limited by the one relation that the value of their determinant is unity. It will also be proved that this system of q^2-1 operators is the system of annihilators of absolute pure q-ary reciprocants, *i.e.*, that, as in the cases of $q = 2$, $q = 3$, such reciprocants are, with a reservation which will appear, identical with absolute differential invariants of the linear group in which generality of coefficients is limited by the one relation just stated.

13. We proceed to consider the transformation of operators linear in symbols of partial differentiation with regard to the partial derivatives $y_{rst\ldots}$ of y with regard to $x_1, x_2, \ldots, x_{q-1}$, consequent upon the q general linear equations of transformation of the q variables

$$x_i = a_{i1}x_1' + a_{i2}x_2' + \ldots + a_{i.q-1}x_{q-1}' + a_{iq}y' + c_i \quad (i = 1, 2, \ldots, q-1),$$

$$y = a_{q1}x_1' + a_{q2}x_2' + \ldots + a_{q.q-1}x_{q-1}' + a_{qq}y' + c_q.$$

Let $\xi_1, \xi_2, \ldots, \xi_{q-1}$ be actual finite increments of $x_1, x_2, \ldots, x_{q-1}$, and η the actual consequent increment of y, given as an expansion in terms of the ξs by Taylor's theorem, as in § 11. Let $\xi_1', \xi_2', \ldots, \xi_{q-1}', \eta'$ be the corresponding increments of the new variables. Then

$$\xi_i = a_{i1}\xi_1' + a_{i2}\xi_2' + \ldots + a_{i.q-1}\xi_{q-1}' + a_{iq}\eta' \quad (i = 1, 2, \ldots, q-1),$$

$$\eta = a_{q1}\xi' + a_{q2}\xi_2' + \ldots + a_{q.q-1}\xi_{q-1}' + a_{qq}\eta';$$

which, if Δ denote the determinant of q^2 constituents $|\ a_{ij}\ |$, and if A_{mn} be its first minor corresponding to a_{mn}, may also be written

$$\Delta\xi_i' = A_{1i}\xi_1 + A_{2i}\xi_2 + \ldots + A_{q-1.i}\xi_{q-1} + A_{qi}\eta \quad (i = 1, 2, \ldots, q-1),$$

$$\Delta\eta' = A_{1q}\xi_1 + A_{2q}\xi_2 + \ldots + A_{q-1.q}\xi_{q-1} + A_{qq}\eta.$$

We have also $\eta' = \Sigma y'_{rst\ldots}\xi_1'^r\xi_2'^s\xi_3'^t \ldots \xi_{q-1}'^{r_{q-1}}$,

a summation exactly corresponding to the already written

$$\eta = \Sigma y_{rst\ldots}\xi_1^r\xi_2^s\xi_3 \ldots \xi_{q-1}^{r_{q-1}}.$$

The formulæ of linear transformation of the variables lead to formulæ for the expression of all derivatives $y_{rst\ldots}$ in terms of derivatives $y'_{rst\ldots}$, and conversely. These formulæ express what are called the extensions of the linear transformation. It is unnecessary to wait to prove here the well-known fact that, because the scheme of transformation of the variables is linear, its extensions do not involve the variables explicitly, but only the derivatives. Thus in the accented equivalent of $\dfrac{\partial}{\partial y_{rst\ldots}}$ in terms of symbols of partial differentiation with regard to accented letters there will be no occurrence of any $\dfrac{\partial}{\partial x'}$ or of $\dfrac{\partial}{\partial y'}$.

If, then, we are transforming an operator

$$\Sigma\left(P_r\frac{\partial}{\partial x_r}\right)+Q\frac{\partial}{\partial y}+\Sigma\left(A_{rst\ldots}\frac{\partial}{\partial y_{rst\ldots}}\right),$$

supposed to act on a function f of $x_1, x_2, \ldots, x_{q-1}, y$, and the derivatives $y_{rst\ldots}$, into its equivalent in form suitable for action on the equivalent function F of the accented variables and derivatives, we may substitute for the first part

$$\Sigma\left(P_r\frac{\partial}{\partial x_r}\right)+Q\frac{\partial}{\partial y}$$

from the formulæ

$$\Delta\frac{\partial}{\partial x_i}=A_{i1}\frac{\partial}{\partial x'_1}+A_{i2}\frac{\partial}{\partial x'_2}+\ldots+A_{i.q-1}\frac{\partial}{\partial x'_{q-1}}+A_{iq}\frac{\partial}{\partial y'}$$
$$(i=1, 2, \ldots, q-1),$$

$$\Delta\frac{\partial}{\partial y}=A_{q1}\frac{\partial}{\partial x'_1}+A_{q2}\frac{\partial}{\partial x'_2}+\ldots+A_{q.q-1}\frac{\partial}{\partial x'_{q-1}}+A_{qq}\frac{\partial}{\partial y'};$$

and transform the remaining part exactly as if it acted on a function of the derivatives only.

14. Now, in the q equations of transformation of the last article for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}, \eta'$, suppose η replaced by its expansion in terms of the ξs with coefficients like $y_{rst\ldots}$, and η' replaced by its similar expansion in terms of the ξ's. We have then q equations connecting the $q-1$ quantities ξ, the $q-1$ quantities ξ', the derivatives $y_{rst\ldots}$, and the derivatives $y'_{rst\ldots}$. As the one relation supposed to exist among y and $x_1, x_2, \ldots, x_{q-1}$ is of perfectly arbitrary

form, and may be varied as we please, we may consider the derivatives $y_{rst\ldots}$ as an infinite number of quite independent variables. Also $\xi_1, \xi_2, \ldots, \xi_{q-1}$ are quite independent of these derivatives and of one another. The other quantities involved in the equations we are considering are dependent variables; viz., $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ are functions of $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and η, which is a function of the others and the derivatives given by its expansion, and the accented derivatives $y'_{rst\ldots}$ are functions of the unaccented derivatives determined by the extensions of the linear transformation. The equations which we are regarding, with η and η' replaced in them by their expansions, are in appearance q equations for the determination of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ in terms of $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and the derivatives. But $q-1$ equations suffice for the purpose. Hence any one of the q must be a consequence of the rest, and the expressions for accented derivatives in terms of unaccented.

Let $y_{rst\ldots}$ alone among the quantities now regarded as independent variables receive an infinitesimal increment $\delta y_{rst\ldots}$, and let $\delta\xi'_1, \delta\xi'_2, \ldots, \delta\xi'_{q-1}$ be the consequent increments of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$.

It is most convenient to regard the q linear equations connecting the ξs and η with the ξ's and η' as before us in their second forms—those which express accented letters in terms of unaccented. These give us at once

$$\Delta . \delta\xi'_i = A_{qi}\xi'^r_1\xi'^s_2\xi'^t_3\ldots\delta y_{rst\ldots} \quad (i = 1, 2, \ldots, q-1)$$

and

$$\Delta\left\{\frac{\partial\eta'}{\partial\xi'_1}\delta\xi'_1 + \frac{\partial\eta'}{\partial\xi'_2}\delta\xi'_2 + \ldots + \frac{\partial\eta'}{\partial\xi'_{q-1}}\delta\xi'_{q-1} + \sum_{\rho,\sigma,\tau,\ldots}\left[\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\xi'^\rho_1\xi'^\sigma_2\xi'^\tau_3\ldots\right]\delta y_{rst\ldots}\right\}$$

$$= A_{qq}\xi'^r_1\xi'^s_2\xi'^t_3\ldots\delta y_{rst\ldots},$$

where the summation with regard to $\rho, \sigma, \tau, \ldots$ covers all zero and positive integral values, such that $\rho+\sigma+\tau+\ldots \nleq 1$. The fact that the first $q-1$ of these equations must lead to the last gives us at once the identity

$$\sum_{\rho,\sigma,\tau,\ldots}\left\{\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\xi'^\rho_1\xi'^\sigma_2\xi'^\tau_3\ldots\right\} \ldots \qquad \ldots$$

$$= \Delta^{-1}\left\{A_{qq} - A_{q1}\frac{\partial\eta'}{\partial\xi'_1} - A_{q2}\frac{\partial\eta'}{\partial\xi'_2} - \ldots - A_{q.q-1}\frac{\partial\eta'}{\partial\xi'_{q-1}}\right\}\xi'^r_1\xi'^s_2\xi'^t_3\ldots.$$

Here Δ is the determinant $|\ a_{ij}\ |$ of the q^2 coefficients in the linear

expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$, and each A_{rs} is a first minor $\frac{\partial \Delta}{\partial a_{rs}}$ of that determinant.

Had we in finding this identity proceeded by consideration of the linear expressions for unaccented in terms of accented letters instead of *vice versa*, the right-hand side would in the first place have presented itself in the form

$$\Delta^{-1} \frac{d(\xi_1, \xi_2, \ldots, \xi_{q-1})}{d(\xi'_1, \xi'_2, \ldots, \xi'_{q-1})} \xi'^{r} \xi'^{s} \xi'^{t} \ldots,$$

where, in the Jacobian functional determinant,

$$\frac{d\xi_r}{d\xi'_s} \text{ means } \frac{\partial \xi_r}{\partial \xi'_s} + \frac{\partial \xi_r}{\partial \eta'} \frac{\partial \eta'}{\partial \xi'_s}, \quad i.e., \; a_{rs} + a_{rq} \frac{\partial \eta'}{\partial \xi'_s}.$$

This latter form, whose identity with the former is easily verified, is occasionally the most convenient for the expression of results.

Now suppose the right-hand member of the identity arrived at to be expanded in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ by means of the linear expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and the expansion for η'. The various coefficients in the development obtained must be severally equal to the corresponding coefficients in the left-hand expansion.

We thus have expressions in terms of accented derivatives of y' for all partial derivatives $\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}$; and these must be those which would be obtained from the extensions of the formulæ of linear transformation, were they actually exhibited.

But let us use otherwise the fact of the absolute identity of the two expansions before us. We may substitute for every product $\xi'^{\rho}_1 \xi'^{\sigma}_2 \xi'^{\tau}_3 \ldots$ any quantity or symbol we please in both of them, and the identity will still remain one. Let us then substitute for them according to the symbolism explained in § 11; *i.e.*, for $\xi'^{\rho}_1 \xi'^{\sigma}_2 \xi'^{\tau}_3 \ldots$ on both sides the corresponding $\frac{\partial}{\partial y'_{\rho\sigma\tau\ldots}}$, the supposition being that it is acting on a function of the derivatives $y'_{\rho\sigma\tau\ldots}$ or some of them, and it may be also of $x'_1, x'_2, \ldots, x'_{q-1}, y'$. The left-hand expansion then becomes

$$\sum_{\rho, \sigma, \tau, \ldots} \left\{ \frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}} \frac{\partial}{\partial y'_{\rho\sigma\tau\ldots}} \right\},$$

i.e., simply

$$\frac{\partial}{\partial y_{rst\ldots}},$$

regarded as operating on the equivalent function of derivatives $y_{rst\ldots}$, and it may be also of $x_1, x_2, \ldots, x_{q-1}, y$.

Accordingly, the transformation of operators of partial differentiation $\dfrac{\partial}{\partial y_{rst\ldots}}$ is before us, and is expressed by the rule: Replace such an operator symbolically by the corresponding product $\xi_1^r \xi_2^s \xi_3^t \ldots$; multiply it by

$$\Delta^{-1}\left\{ A_{qq} - A_{q1}\frac{\partial \eta'}{\partial \xi_1'} - A_{q2}\frac{\partial \eta'}{\partial \xi_2'} - \ldots - A_{q,q-1}\frac{\partial \eta'}{\partial \xi_{q-1}'} \right\};$$

express the result in terms of $\xi_1', \xi_2', \ldots, \xi_{q-1}'$ by the linear formulæ and the expansion for η'; expand in powers and products of powers of $\xi_1', \xi_2', \ldots, \xi_{q-1}'$; and, finally, for each power or product $\xi_1'^{\rho} \xi_2'^{\sigma} \xi_3'^{\tau} \ldots$ which occurs in the expansion, write the corresponding operator of partial differentiation $\dfrac{\partial}{\partial y'_{\rho\sigma\tau\ldots}}$.

Moreover, the transformation of any linear partial differential operator (see § 13),

$$\Sigma\left(A_{rst\ldots}\frac{\partial}{\partial y_{rst\ldots}} \right),$$

is at once effected in like manner. Write it symbolically

$$\Sigma\left(A_{rst\ldots} \xi_1^r \xi_2^s \xi_3^t \ldots \right);$$

multiply it throughout by the factor just written; and complete the process of expansion and interpretation of symbolic products as before.

15. We will now consider in particular the transformation of the operators of § 12. We may deal with such operators together by considering the operator

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \left\{ \lambda_q + \lambda_1 \frac{\partial \eta}{\partial \xi_1} + \lambda_2 \frac{\partial \eta}{\partial \xi_2} + \ldots + \lambda_{q-1}\frac{\partial \eta}{\partial \xi_{q-1}} \right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}, \lambda_q$ are q arbitrary constants. The symbolism is that explained in § 11.

By immediate application of the rule just arrived at, the transformation of this operator is symbolically

$$\Delta^{-1}\left\{ A_{qq} - A_{q1}\frac{\partial \eta'}{\partial \xi_1'} - A_{q2}\frac{\partial \eta'}{\partial \xi_2'} - \ldots - A_{q,q-1}\frac{\partial \eta'}{\partial \xi_{q-1}'} \right\}$$
$$\times \left\{ \lambda_q + \lambda_1 \frac{\partial \eta}{\partial \xi_1} + \lambda_2 \frac{\partial \eta}{\partial \xi_2} + \ldots + \lambda_{q-1}\frac{\partial \eta}{\partial \xi_{q-1}} \right\} \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m;$$

but this may be written in better form.

Identification of the Pfaffian equations

$$d\eta - \frac{\partial \eta}{\partial \xi_1} d\xi_1 - \frac{\partial \eta}{\partial \xi_2} d\xi_2 - \ldots - \frac{\partial \eta}{\partial \xi_{q-1}} d\xi_{q-1} = 0,$$

$$d\eta' - \frac{\partial \eta'}{\partial \xi_1'} d\xi_1' - \frac{\partial \eta'}{\partial \xi_2'} d\xi_2' - \ldots - \frac{\partial \eta'}{\partial \xi_{q-1}'} d\xi_{q-1}' = 0,$$

where the ξs and η are actual quantities linearly expressed in terms of the ξ's and η', gives

$$\frac{a_{qq} - a_{1q}\dfrac{\partial \eta}{\partial \xi_1} - a_{2q}\dfrac{\partial \eta}{\partial \xi_2} - \ldots - a_{q-1.q}\dfrac{\partial \eta}{\partial \xi_{q-1}}}{1}$$

$$= \frac{a_{qi} - a_{1i}\dfrac{\partial \eta}{\partial \xi_1} - a_{2i}\dfrac{\partial \eta}{\partial \xi_2} - \ldots - a_{q-1.i}\dfrac{\partial \eta}{\partial\ _{q-1}}}{-\dfrac{\partial \eta'}{\partial \xi_i'}} \quad (i = 1, 2, \ldots, q-1)$$

$$= \frac{-\Delta \dfrac{\partial \eta}{\partial \xi_i}}{A_{iq} - A_{i1}\dfrac{\partial \eta'}{\partial \xi_1'} - A_{i2}\dfrac{\partial \eta'}{\partial \xi_2'} - \ldots - A_{i.q-1}\dfrac{\partial \eta'}{\partial \xi_{q-1}'}} \quad (i = 1, 2, \ldots, q-1)$$

$$= \frac{\Delta}{A_{qq} - A_{q1}\dfrac{\partial \eta'}{\partial \xi_1'} - A_{q2}\dfrac{\partial \eta'}{\partial \xi_2'} - \ldots - A_{q.q-1}\dfrac{\partial \eta'}{\partial \xi_{q-1}'}}.$$

Thus

$$\left\{ A_{qq} - A_{q1}\frac{\partial \eta'}{\partial \xi_1'} - A_{q2}\frac{\partial \eta'}{\partial \xi_2'} - \ldots - A_{q.q-1}\frac{\partial \eta'}{\partial \xi_{q-1}'} \right\} \frac{\partial \eta}{\partial \xi_i}$$

$$= - \left\{ A_{iq} - A_{i1}\frac{\partial \eta'}{\partial \xi_1'} - A_{i2}\frac{\partial \eta'}{\partial \xi_2'} - \ldots - A_{i.q-1}\frac{\partial \eta'}{\partial \xi_{q-1}'} \right\}.$$

Applying this fact for each value of i, we see that the transformed operator above may be written

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \left\{ \lambda_q' + \lambda_1' \frac{\partial \eta'}{\partial \xi_1'} + \lambda_2' \frac{\partial \eta'}{\partial \xi_2'} + \ldots + \lambda_{q-1}' \frac{\partial \eta'}{\partial \xi_{q-1}'} \right\},$$

where

$$\Delta \lambda_i' = A_{1i}\lambda_1 + A_{2i}\lambda_2 + \ldots + A_{q-1.i}\lambda_{q-1} + A_{qi}(-\lambda_q) \quad (i = 1, 2, \ldots, q-1),$$

$$\Delta(-\lambda_q') = A_{1q}\lambda_1 + A_{2q}\lambda_2 + \ldots + A_{q-1.q}\lambda_{q-1} + A_{qq}(-\lambda_q),$$

or, more simply,

$$\lambda_i = a_{i1}\lambda_1' + a_{i2}\lambda_2' + \ldots + a_{i,q-1}\lambda_{q-1}' + a_{iq}(-\lambda_q') \quad (i = 1,2,\ldots,q-1),$$

$$-\lambda_q = a_{q1}\lambda_1' + a_{q2}\lambda_2' + \ldots + a_{q,q-1}\lambda_{q-1}' + a_{qq}(-\lambda_q'),$$

so that $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}, -\lambda_q$ express in terms of $\lambda_1', \lambda_2', \ldots, \lambda_{q-1}', \lambda_q'$, as do quantities cogredient with $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$. We shall exemplify the use of this fact later.

For the present our object is to draw a conclusion from the form of the transformed operator without paying special attention to this connexion between the λs and λ's. We have, in particular, that the transformations of

$$\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m \quad \text{and} \quad \xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m\frac{\partial\eta}{\partial\xi_r}$$

are of the above form, *i.e.*, since we can express $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ linearly in terms of $\xi_1', \xi_2', \ldots, \xi_{q-1}', \eta'$, are linear functions of operators of the same two forms as themselves with different values of $n_1, n_2, \ldots, n_{q-1}, m$, and r. The sum $\Sigma n+m$ is, however, the same for each part of the transformed operators as in the operators subjected to transformation, in virtue of the linearity of the expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$. Consequently, an operator of any order $\Sigma n+m$ transforms into a linear function with constant coefficients of operators of the same order $\Sigma n+m$.

In particular, the complete system of operators of the first order (§ 12) transforms into the complete system of operators of the first order, a fact in accord with the next article.

16. I will now utilize the method of transformation before us to prove that, as stated earlier, the q^2 independent operators of the first order are the operators appertaining to the q^2 independent infinitesimal substitutions of the group with q^2 parameters, which is constituted by the extensions of the general linear group with $q(q+1)$ parameters; so that the writing down of explicit forms of these operators of infinitesimal substitution to any extent is reduced to mere multiplication of multinomials, and, indeed, to the mere squaring of one multinomial η.

Consider, first, a substitution which leaves $x_1, x_2, \ldots, x_{q-1}$ unaltered, but replaces y by

$$a_1x_1 + a_2x_2 + \ldots + a_{q-1}x_{q-1} + (1+a_q)\,y + \gamma,$$

where $a_1, a_2, \ldots, a_q, \gamma$ are infinitesimal constants. Here γ does not

affect the derivatives $y_{rst...}$. The increment which it gives to a function f of variables and derivatives is only $\gamma \frac{\partial}{\partial y} f$.

Again, $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ affect each only one, a first, derivative. Thus, for instance, α_1 alters only $y_{100...}$, changing it into $y_{100...} + \alpha_1$. Thus the $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ increments of f are respectively

$$\alpha_1\left(x_1\frac{\partial}{\partial y} + \frac{\partial}{\partial y_{100...}}\right)f, \quad \alpha_2\left(x_2\frac{\partial}{\partial y} + \frac{\partial}{\partial y_{010...}}\right)f, \quad \alpha_3\left(x_3\frac{\partial}{\partial y} + \frac{\partial}{\partial y_{001...}}\right), \quad \ldots,$$

i.e., $$\alpha_1\left(x_1\frac{\partial}{\partial y} + \xi_1\right)f, \quad \alpha_2\left(x_2\frac{\partial}{\partial y} + \xi_2\right)f, \quad \ldots, \quad \alpha_{q-1}\left(x_{q-1}\frac{\partial}{\partial y} + \xi_{q-1}\right)f.$$

Once more, the existence of α_q changes $y_{rst...}$ into $(1+\alpha_q)\, y_{rst...}$. Its effect is then to give f the increment

$$\alpha_q\left(y\frac{\partial}{\partial y} + \Sigma y_{rst...}\frac{\partial}{\partial y_{rst..}}\right)f,$$

i.e., $$\alpha_q\left(y\frac{\partial}{\partial y} + \eta\right)f.$$

Again, consider a substitution which leaves y and $x_2, x_3, \ldots, x_{q-1}$ unaltered, but replaces x_1 by

$$(1+\beta_1)\, x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_{q-1} x_{q-1} + \beta_q y + \gamma',$$

where the βs and γ' are infinitesimal constants. The consequent increment of f is in just the same way what, in a notation which regards $y, x_2, x_3, \ldots, x_{q-1}$ as independent variables and x_1 as dependent, would be written

$$\left\{\gamma'\frac{\partial}{\partial x_1} + \beta_2\left(x_2\frac{\partial}{\partial x_1} + \xi_2\right) + \beta_3\left(x_3\frac{\partial}{\partial x_1} + \xi_3\right) + \ldots\right.$$

$$\left. \ldots + \beta_{q-1}\left(x_{q-1}\frac{\partial}{\partial x_1} + \xi_{q-1}\right) + \beta_q\left(y\frac{\partial}{\partial x_1} + \eta\right) + \beta_1\left(x_1\frac{\partial}{\partial x_1} + \xi_1\right)\right\} f;$$

but we have here to express the symbolic operators by their equivalents when, as in fact, y is dependent and $x_1, x_2, \ldots, x_{q-1}$ independent. In other words, we have to transform the operator as in the last two articles, taking

$$x_2 = x_2', \quad x_3 = x_3', \quad \ldots, \quad x_{q-1} = x_{q-1}', \quad y = y', \quad x_1 = x_1',$$

where, on the left, x_1 is dependent, and, on the right, y'; and then remove accents.

Now, here, $\Delta = \begin{vmatrix} 1, & 0, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & 0, & \ldots, & 0, & 0 \\ 0, & 0, & 1, & \ldots, & 0, & 0 \\ \ldots & & \ldots & & \ldots & \ldots \\ 0, & 0, & 0, & \ldots, & 0, & 1 \\ 0, & 0, & 0, & \ldots, & 1, & 0 \end{vmatrix} = -1,$

and the other factor which has to be introduced as well as Δ^{-1} may, as we saw in § 14, be calculated as the Jacobian of the old independent Greek letters (in this case $\xi_2, \xi_3, \ldots, \xi_{q-1}, \eta$) with regard to the new independent ones (in this case $\xi'_2, \xi'_3, \ldots, \xi'_{q-1}, \xi'_1$). It is then

$$\begin{vmatrix} 1, & 0, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & 0, & \ldots, & 0, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & 0, & \ldots, & 1, & 0 \\ \dfrac{\partial\eta'}{\partial\xi'_2}, & \dfrac{\partial\eta'}{\partial\xi'_3}, & \dfrac{\partial\eta'}{\partial\xi'_4}, & \ldots, & \dfrac{\partial\eta'}{\partial\xi'_{q-1}}, & \dfrac{\partial\eta'}{\partial\xi'_1} \end{vmatrix} = \frac{\partial\eta'}{\partial\xi'_1}.$$

Thus we have to substitute for $\xi_2, \xi_3, \ldots, \xi_{q-1}, \eta, \xi_1$, their equivalents with y dependent $-\xi_2\dfrac{\partial\eta'}{\partial\xi'_1}, -\xi_3\dfrac{\partial\eta'}{\partial\xi'_1}, \ldots, -\xi_{q-1}\dfrac{\partial\eta'}{\partial\xi'_1}, -\eta\dfrac{\partial\eta'}{\partial\xi'_1}, -\xi_1\dfrac{\partial\eta'}{\partial\xi'_1}$, where $\xi_2, \xi_3, \ldots, \xi_{q-1}, \eta, \xi_1$ are the same as $\xi'_2, \xi'_3, \ldots, \xi'_{q-1}, \eta', \xi'_1$. Doing so, and removing the accents which have been used only for temporary convenience, the aggregate of the second set of infinitesimal increments of f becomes

$$\left\{\gamma'\frac{\partial}{\partial x_1} + \beta_2\left(x_2\frac{\partial}{\partial x_1} - \xi_2\frac{\partial\eta}{\partial\xi_1}\right) + \beta_3\left(x_3\frac{\partial}{\partial x_1} - \xi_3\frac{\partial\eta}{\partial\xi_1}\right) + \ldots\right.$$

$$\left. \ldots + \beta_{q-1}\left(x_{q-1}\frac{\partial}{\partial x_1} - \xi_{q-1}\frac{\partial\eta}{\partial\xi_1}\right) + \beta_q\left(y\frac{\partial}{\partial x_1} - \eta\frac{\partial\eta}{\partial\xi_1}\right) + \beta_1\left(x_1\frac{\partial}{\partial x_1} - \xi_1\frac{\partial\eta}{\partial\xi_1}\right)\right\} f.$$

In these two partial aggregates of infinitesimal substitutions, symbolic operators of all types $\xi_r, \eta, \xi_r\dfrac{\partial\eta}{\partial\xi_r}, \xi_r\dfrac{\partial\eta}{\partial\xi_s}\,(r \neq s), \eta\dfrac{\partial\eta}{\partial\xi_s}$ have been introduced, every one which has occurred, occurring once only; and no operators, having reference to derivatives $y_{rst\ldots}$, of any other kind are present. It is clear that we complete the entire aggregate of infinitesimal substitutions by giving next x_2 its most general infinitesimal increment, keeping $x_1, x_3, \ldots, x_{q-1}, y$ unaltered; then x_3

in like manner, and so on. No new type of operator is thus introduced; but every one of each type, as above, is just once presented. The proof for the case of x_1, which has been given, applies to all the others with alterations of suffixes.

We thus have it clearly before us that the $q(q+1)$ operators appertaining to the independent infinitesimal substitutions of the general linear group and its extensions are

(i.) one of the type $\dfrac{\partial}{\partial y}$,

(ii.) $q-1$ of the type $\dfrac{\partial}{\partial x_r}$,

(iii.) $q-1$ of the type $x_r \dfrac{\partial}{\partial y} + \xi_r$,

(iv.) one of the type $y \dfrac{\partial}{\partial y} + \eta$,

(v.) $q-1$ of the type $x_r \dfrac{\partial}{\partial x_r} - \xi_r \dfrac{\partial \eta}{\partial \xi_r}$,

(vi.) $(q-1)(q-2)$ of the type $x_r \dfrac{\partial}{\partial x_s} - \xi_r \dfrac{\partial \eta}{\partial \xi_s}$ $(r \neq s)$,

(vii.) $q-1$ of the type $y \dfrac{\partial}{\partial x_s} - \eta \dfrac{\partial \eta}{\partial \xi_s}$.

The q^2 operators appertaining to the group which consists of the extensions only are given by (iii.) to (vii.) inclusive with the $\dfrac{\partial}{\partial y}$, $\dfrac{\partial}{\partial x_r}$, $\dfrac{\partial}{\partial x_s}$ parts omitted.

As to the sub-group of the general linear group in which the generality of the constants is limited by the one condition

$$\Delta \equiv | a_{ij} | = 1,$$

the fact of its having one infinitesimal substitution fewer is expressed by the vanishing of the sum of the infinitesimal multipliers of the q operators (iv.) and (v.). The conclusions which follow have been stated in the latter part of § 12.

17. Let us now think of differential invariants, with a view to arriving at a proof that the absolute differential invariants of this

last-mentioned sub-group—an invariant sub-group (*Lie*) or self-conjugate sub-group (*Burnside*) of the general linear group—are identical with absolute pure q-ary reciprocants; or, more precisely, that absolute pure q-ary reciprocants are coextensive with these absolute differential invariants of the sub-group, which are homogeneous or entirely of odd or entirely of even degree.

Notice that it is not a necessity, but a convention, to regard the differential invariants of the sub-group, or the pure reciprocants, as homogeneous,* just as it is not a necessity, but a convention, to regard ordinary invariants,† except those of a single quantic, as homogeneous. The convention is reasonable; for the separate homogeneous parts of a non-homogeneous differential invariant or pure reciprocant are themselves differential invariants or pure reciprocants, just as in the case of ordinary invariants. It will illustrate the usefulness of our present method to prove this for absolute differential invariants of the self-conjugate sub-group.

The following is an equivalence of symbolic operators for the general linear scheme :—

$$\xi_1\frac{\partial\eta}{\partial\xi_1}+\xi_2\frac{\partial\eta}{\partial\xi_2}+\ldots+\xi_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}-\eta\equiv\xi'_1\frac{\partial\eta'}{\partial\xi'_1}+\xi'_2\frac{\partial\eta'}{\partial\xi'_2}+\ldots+\xi'_{q-1}\frac{\partial\eta'}{\partial\xi'_{q-1}}-\eta'.$$

To prove it we apply § 15. The operator which transforms into

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* In the second of my papers on ternary reciprocants the unjustifiable remark was made (*Proc. Lond. Math. Soc.*, Vol. XVIII., p.157) that a ternary reciprocant is *necessarily* homogeneous, and was based on the unjustifiable statement that this is so for ordinary invariants of a system of binary quantics.

† An ordinary q-ary invariant or covariant (non-absolute) is an absolute invariant of a group holohedrically isomorphous with a linear homogeneous group of q variables in which the generality of coefficients is limited by the one relation $\Delta = 1$, the special sub-group of the general linear homogeneous group. For instance, when the variables in $(a_0, a_1, \ldots, a_p)(x, y)^p$ are substituted for, according to this special group, the group of substitutions for the coefficients involves the same independent parameters, and the infinitesimal transformations, namely, in the one case $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$, $y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$, and, in the other,

$$a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\ldots,\quad pa_1\frac{\partial}{\partial a_0}+(p-1)\,a_2\frac{\partial}{\partial a_1}+\ldots,\quad pa_0\frac{\partial}{\partial a_0}+(p-2)\,a_1\frac{\partial}{\partial a_1}+\ldots,$$

have the same composition. An absolute invariant or covariant is an absolute invariant of a group holohedrically isomorphous with the general linear homogeneous group.

$\xi'_r \dfrac{\partial \eta'}{\partial \xi'_r}$ is, by immediate application of that article,

$$\xi'_r \left\{ a_{1r} \frac{\partial \eta}{\partial \xi_1} + a_{2r} \frac{\partial \eta}{\partial \xi_2} + \ldots + a_{q-1,r} \frac{\partial \eta}{\partial \xi_{q-1}} - a_{qr} \right\}$$

$$(r = 1, 2, \ldots, q-1),$$

and that which transforms into $-\eta'$ is

$$\eta' \left\{ a_{1q} \frac{\partial \eta}{\partial \xi_1} + a_{2q} \frac{\partial \eta}{\partial \xi_2} + \ldots + a_{q-1,q} \frac{\partial \eta}{\partial \xi_{q-1}} - a_{qq} \right\};$$

addition of which gives, by means of the linear expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}, \eta'$, the identity which has been written.

From this equivalence, a persistence in form of an operator, it follows that

$$\xi_1 \frac{\partial \eta}{\partial \xi_1} + \xi_2 \frac{\partial \eta}{\partial \xi_2} + \ldots + \xi_{q-1} \frac{\partial \eta}{\partial \xi_{q-1}} - \eta$$

is an operator which, acting on an absolute differential invariant of a linear group, produces another absolute differential invariant. It annihilates one of the general linear group, but not, as a rule, one of the sub-group.

Now absolute differential invariants of the sub-group we are considering have $\eta + \xi_1 \dfrac{\partial \eta}{\partial \xi_1}$, $\eta + \xi_2 \dfrac{\partial \eta}{\partial \xi_2}$, ... for annihilators (§ 12). Hence the effect on any such of $\xi_r \dfrac{\partial \eta}{\partial \xi_r}$ (for any r) is the same as that of $-\eta$. It follows that $-q\eta$, and consequently η, is an operator which generates absolute differential invariants of the sub-group from others. Now the effect of η is to multiply any homogeneous function by its degree. If, then,

$$H_i + H_j + H_k + \ldots$$

is an absolute differential invariant with parts of degrees $i, j, k, \ldots$, it follows in succession that

$$iH_i + jH_j + kH_k + \ldots,$$

$$i^2 H_i + j^2 H_j + k^2 H_k + \ldots,$$

$$\&c., \qquad \&c.,$$

are also absolute differential invariants; and, consequently, that H_i, H_j, H_k, ... separately are. We lose, then, no generality, as to complete systems, if we confine attention to differential invariants which are homogeneous.

Let us now consider what a homogeneous absolute differential invariant of our sub-group becomes when a linear substitution is applied for which $\Delta = -1$ instead of $+1$. Such a substitution may be effected by two stages: (1) a change of sign in y, and (2) a substitution of determinant $+1$. The first stage does not alter the differential invariant if its degree be even, but changes its sign if its degree be odd. The second stage makes no alteration in it, being merely the performance of a substitution for which it is invariant. Thus a function of derivatives which does not contain parts of both odd and even degrees, and which is an absolute differential invariant for a linear group for which $\Delta = +1$, is, at most, changed in sign by a substitution for which $\Delta = -1$.

18. In my paper "On Ternary and n-ary Reciprocants" (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 191), the only one in which cases of n (*i.e.*, q) > 3 have been introduced, I gave only a provisional definition of a q-ary reciprocant, expressing no confidence that the best form of presentation was given. It now seems better to adopt the following definition for the class of q-ary reciprocants which is of importance in the present connexion. A function of the derivatives of y with regard to $x_1, x_2, \ldots, x_r, \ldots, x_{q-1}$ is called an absolute q-ary reciprocant when it is equal, but for a constant factor, to the same function of the derivatives of x_r with regard to $x_1, x_2, \ldots, y, \ldots, x_{q-1}$, for each of the $q-1$ values of r. Here y occupies the old position of x_r in the series of independent variables. The reciprocant is called *pure* when first derivatives do not occur in it. All this is in strict accord with Sylvester's definition of ordinary reciprocants. The constant factor is either ± 1, as the conclusions which follow will establish.

We notice, first, that any absolute differential invariant, which does not contain parts of both odd and even degrees, of the linear group with generality of coefficients limited by the one relation $\Delta = 1$, is an absolute q-ary reciprocant. For the substitution which effects a change of independent variable as above is (as in § 16) one for which $\Delta = -1$, and the effect of this on the differential invariant is, at most, to change its sign, by the preceding article. One which contains parts both of odd and of even degrees is not a q-ary reciprocant, but the sum of two such of opposite characters.

We have, further, to prove conversely that any absolute pure q-ary reciprocant is an absolute differential invariant for the special linear group. We shall see this by showing that it has the full system of annihilators of absolute differential invariants of the group.

The property of absolute pure q-ary reciprocants which suffices to give the entire conclusion is that any one, and all its equivalents arising from interchanges of the dependent with one of the independent variables, are free from first derivatives.

That the reciprocant R_y itself (y dependent) is free from first derivatives is expressed by saying that it has the $q-1$ annihilators

$$\xi_1, \xi_2, \xi_3, \ldots, \xi_{q-1}.$$

That its equivalent with x_1 dependent and y occupying the place of x_1 is free from first derivatives is, in like manner, expressed by saying that it has $q-1$ annihilators which, were x_1 dependent variable, would be

$$\eta, \xi_2, \xi_3, \ldots, \xi_{q-1}.$$

Now, as in § 16, these annihilators, expressed in the notation which has meaning when y is dependent, are, respectively,

$$-\eta\frac{\partial\eta}{\partial\xi_1}, \quad -\xi_2\frac{\partial\eta}{\partial\xi_1}, \quad -\xi_3\frac{\partial\eta}{\partial\xi_1}, \quad \ldots, \quad -\xi_{q-1}\frac{\partial\eta}{\partial\xi_1}.$$

In like manner, expressing that the equivalents of R_y with $x_2, x_3, \ldots, x_{q-1}$ in succession as the one dependent variable are free from first derivatives, we get $q-2$ other sets of $q-1$ annihilators of R_y, but no new types are introduced. So far, then, the information is that R_y has

(i.) $q-1$ annihilators $\xi_1, \xi_2, \ldots, \xi_{q-1}$,

(ii.) $q-1$ annihilators $\eta\dfrac{\partial\eta}{\partial\xi_r}$ $(r = 1, 2, \ldots, q-1)$,

(iii.) $(q-1)(q-2)$ annihilators $\xi_r\dfrac{\partial\eta}{\partial\xi_s}$ $(r \neq s;\ r, s = 1, 2, \ldots, q-1)$,

i.e., together q^2-q annihilators.

But each of the facts (ii.) really gives two facts of annihilation independent of one another and of (i.) and (iii.), in consequence of R_y being free from first derivatives. Consider, for instance, the annihilation by $\eta\dfrac{\partial\eta}{\partial\xi_1}$. The terms in the expansion of $\eta\dfrac{\partial\eta}{\partial\xi_1}$ which

involve the first derivative $y_{100\ldots}$ in their coefficients must by themselves annihilate R_y, as otherwise operation with them would produce terms in $\eta \frac{\partial \eta}{\partial \xi_1} R_y$ against which no other terms could cancel. Now, as η only involves $y_{100\ldots}$ in the term $y_{100\ldots}\xi_1$, these terms in $\eta \frac{\partial \eta}{\partial \xi_1}$ are $y_{100\ldots}\left(\xi_1 \frac{\partial \eta}{\partial \xi_1} + \eta\right)$. Similarly as to $\eta \frac{\partial \eta}{\partial \xi_r}$ in general. Thus, besides the $q^2 - q$ annihilators above classified, R_y must also have

(iv.) the $q-1$ annihilators $\xi_r \frac{\partial \eta}{\partial \xi_r} + \eta$, $(r = 1, 2, \ldots, q-1)$.

Consequently, our supposed absolute pure q-ary reciprocant R_y has for annihilators the full system of $q^2 - 1$ operators which belong to the infinitesimal substitutions of the group of extensions of the linear group with generality of coefficients limited by the one relation $\Delta = 1$. R_y is then an absolute differential invariant of that special linear group.

Accordingly, the statement that absolute pure q-ary reciprocants are identical with those absolute differential invariants which do not contain parts of both odd and even degrees of the special linear group is correct.

Thursday, April 7th, 1898.

Dr. E. W. HOBSON, F.R.S., Vice-President, in the Chair.

There being only five members present, and so no quorum, no meeting could be held.

The Chairman (provisionally) communicated the following papers:—

An Essay towards the Generating Functions of Ternariants: Professor Forsyth.

On Systems of One-Vectors in Space of n Dimensions: Mr. W. H. Young.

Zeroes of the Bessel Functions: Mr. H. M. Macdonald.

The following presents were received for the Library :—

"Proceedings of the Royal Society," Vol. LXII., Nos. 386, 387.

"Year-Book of the Royal Society," 1896-7, No. 1.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," V., VI ; 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 3 ; Leipzig, 1898.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. IV., No. 6 ; New York, March, 1898.

"Science Abstracts," Vol. I., Pts. 1, 2 ; London, Taylor & Francis, 1898.

Rayet, G.—"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde" de Juin 1894 à Mai 1895 ; Juin 1895 à Mai 1896, 8vo; Bordeaux, 1895-6.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. IV., Fasc. 2 ; Napoli, 1898.

"Mémoires de la Société des Sciences Physiques et Natureles de Bordeaux," Tome I., Cahier 1, 2, 1895-6 ; Tome II , Cahier 1, 2, 1896. "Procès-Verbaux de Séances de la Société des Sciences Physiques et Naturelles de Bordeaux," Année 1894-5 ; 1895-6.

"Comptes Rendus de la Section de Physique et de Chimie de Varsovie," 1895-6 ; 1897.

"Comptes Rendus de la Section Biologique, 1895-6," et "Procès-Verbaux des Réunions Générales," Année 7 ; Varsovie.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 5 ; Roma, 1898.

"Educational Times," April, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome XII., Fasc. 1, 2 ; 1898.

"Rendiconti del Circolo Matematico di Palermo," Tome XII., Fasc. 1, 2 ; February, 1898.

"Annals of Mathematics," Vol. XII., No. 1 ; Virginia, February, 1898.

"Indian Engineering," Vol. XXIII., Nos. 8, 9, 10, Feb. 19-March 5, 1898.

Nova Acta :—

"Beiträge zur Geschichte der Trigonometrie," von A. v. Braunmühl, Bd. LXXI., 1 ; Halle, 1897.

"Nassîr Eddîn Tûsi und Regiomontan," Bd. LXXI., 2, von A. v. Braunmühl ; Halle, 1897.

"Zur Geschichte der Geometrie mit constanter Zirkelöffnung," Bd. LXXI., 3, von W. M. Kotta.

("Abh. der Kaiserl. Leop.-Carol. Deutschen Akad. der Naturforschen.")

"On the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$, with extended Tables of Values," by Jas. Burgess, Edinburgh (Offprint from the "Transactions of the Royal Society of Edinburgh," Vol. XXXIX., Pt. II., No. 9).

On Systems of One-Vectors in Space of n Dimensions. *By* W. H. Young. Received March 26, 1898. Provisionally communicated April 7, 1898.

In the present paper a system of one-vectors* in space of n dimensions is reduced to a normal form.

The space in question is supposed to be flat, *i.e.*, of zero curvature.

I use S_k to denote a flat space of k dimensions contained in the whole space S_n.

The normal form in S_{2m-1} consists of m one-vectors, the line of action of one being arbitrary. In S_{2m} the normal form again consists of m one-vectors, lying, however, in a covariant S_{2m-1}.

In the course of the discussion certain other properties of the system are obtained. The whole subject is closely connected with the theory of linear complexes in space of n dimensions. I here confine myself to giving a geometrical definition of line coordinates in such a space, and a simple geometrical deduction of the known quadratic equations between them.

It is well known that in S_n we can construct, in an infinite number of ways, a fundamental $(n+1)$-pyramid which does not lie in any space of lower dimensions.

THEOREM I. *Any one-vector through an arbitrary point O may be replaced by n one-vectors along lines joining O to the n points of any fundamental n-pyramid lying in any S_{n-1} not passing through O.*

These n points together with O define a fundamental $(n+1)$-pyramid in the S_n.

Take the plane through the line of action of the one-vector and one of the edges meeting at O. This plane meets the S_{n-1} determined by the remaining $(n-1)$ edges through O in a straight line. We can then, by the parallelogram law, replace our one-vector by two components, one along the first edge, and one in the S_{n-1}. In this

* The reader may, if he please, substitute the word "force" for "one-vector," throughout the present paper, which is, in part, introductory to one on vectors of a more general nature. In a paper entitled "Sulla Statica dei Corpi Rigidi nello Spazio a Quattro Dimensioni," *Giornale di Battaglini*, XXXIV., 1896, some of the properties here given are discussed for four dimensions by De Francesco.

S_{n-1} the n vertices of the original $(n+1)$-pyramid determine a fundamental n-pyramid, and the component of our one-vector in this S_{n-1} goes through one of the vertices, viz., O. Thus, by induction, the theorem follows, it being certainly true when $n = 2$.

The above tacitly assumes that the one-vector does not lie in any S_k of the fundamental $(n+1)$-pyramid. If this be the case, we only have to effect the reduction for the corresponding $(k+1)$-pyramid, the components corresponding to the remaining edges through O vanishing.

THEOREM II. *Any one-vector may be replaced by one-vectors along the edges of the fundamental $(n+1)$-pyramid.*

Let us assume that the line of action of the one-vector does not lie in any S_k of the fundamental $(n+1)$-pyramid. It must intersect at least one of the S_{n-1}'s of the pyramid in a point at a finite distance. Join this point of intersection to the vertex S_0 of the pyramid opposite to this S_{n-1}. The plane through the one-vector and this line will meet the S_{n-1} in question in a straight line through the same point of intersection. Hence, by the parallelogram law, the one-vector may be replaced by two one-vectors, one through the S_0 and the other in the S_{n-1}. By Theorem I. the former component may be replaced by components along the edges through the S_0. The possibility of the reduction then depends on replacing the one-vectors in the S_{n-1} by components along the edges of the fundamental $(n+1)$-pyramid which lie in that S_{n-1}. But these are the edges of an n-pyramid which is a fundamental one in that S_{n-1}. The required result follows by induction, for it obviously holds when $n = 1$.

THEOREM III. *A given system of one-vectors may be replaced in one and only one way by one-vectors along the edges of a given fundamental $(n+1)$-pyramid.*

The possibility of this reduction follows from Theorem II.

To prove the uniqueness, we remark that, if two modes of reduction are possible, reversing one system so obtained and combining with the other, we obtain a system in equilibrium.* This system of one-vectors along the edges may be divided into two classes, viz., those through an arbitrarily chosen vertex and those in the opposite S_{n-1} of the fundamental $(n+1)$-pyramid. Unless the former all vanish, they can, by repeated application of the parallelogram law, be reduced

* A system is said to balance, or to be in equilibrium, when, by introducing suitable pairs of equal and opposite one-vectors and by repeated application of the parallelogram law, it can be reduced to two equal and opposite one-vectors.

to a single one-vector through the chosen vertex. This one-vector, lying outside the S_{n-1}, cannot be replaced by, and therefore also cannot balance, a system of one-vectors in it. The theorem now follows by induction.

Cor. I. *A system of one-vectors has* $\frac{1}{2}(n+1)\,n$ *"coordinates," viz., the equivalent one-vectors along the edges of the fundamental* $(n+1)$*-pyramid. There are, moreover,* $\frac{1}{2}(n+1)\,n$ *necessary and sufficient conditions of equilibrium, viz., each coordinate must vanish.*

Cor. II. *There are* $\frac{1}{2}(n-1)(n-2)$ *necessary and sufficient conditions that a given system of one-vectors should be equivalent to a single one-vector.*

This is evident if we reflect that there are ∞^{2n-2} straight lines in S_n, and that therefore a straight line has $(2n-2)$ independent coordinates; a single one-vector therefore $(2n-1)$. The required number is got by subtracting this number from the number of coordinates of a general system.

Cor. III. *Any system of one-vectors in an* S_n *may be replaced by a one-vector through any arbitrary point O, and a system of one-vectors in an arbitrary* S_{n-1} *which does not pass through O.*

In a special case the one-vector may of course vanish, or the system in the S_{n-1} be in equilibrium.

We now proceed to generalize certain known theorems about systems of one-vectors in S_3. It will be convenient to consider first the cases of S_4 and S_5.

Theorem IV. *Any system of one-vectors in* S_4 *is equivalent to two one-vectors, and has as covariant the* S_3 *containing them.*

By Cor. III. to Theorem III. such a system is equivalent to a one-vector through an arbitrary point O and a system of one-vectors in an S_3. This latter system, if not reducible to a single one-vector, is, as is well known, equivalent to two one-vectors, one of which may be chosen to act along any line we please in the S_3. Take it to act along a line through the point in which the one-vector through O meets the S_3. Then it may be compounded with the latter one-vector, and thus the system is reducible to two one-vectors, which, in general, do not intersect.

These two one-vectors may be replaced by any other suitable pair in the S_3 determined by them; viz., the line of action of one of the two one-vectors may be chosen arbitrarily in the S_3, the line of action of the other and the magnitude of both one-vectors being then determined. The S_3 itself is a covariant of the system.

In fact, if it were possible by reducing in another way to get a second pair lying in a different S_3, we could reverse this pair and obtain with the first pair a system in equilibrium. Since, however, two S_3's intersect in an S_2, we can arrange that one one-vector of our second pair should lie in this S_2, and the other accordingly not. We should then have a single one-vector not in the first S_3, balancing one-vectors lying in it, which is obviously absurd.

THEOREM V. *a. Any system of one-vectors in S_5 is equivalent to three one-vectors, one of which may be chosen to act through any point we choose.*

The reduction to three one-vectors follows from the previous theorem by means of Cor. III. to Theorem III. One of the one-vectors passes by construction through an arbitrary point O. If the S_3 determined by any two one-vectors be intersected by the line of action of the third, the three one-vectors would lie in an S_4. That this would be a special case is obvious, since we might choose as our system three straight lines not lying in an S_4. Omitting for the present the discussion of special cases, we may assume in what follows that the three one-vectors do not lie in an S_4.

THEOREM V. *b. Any arbitrary line being chosen, one of the three one-vectors can be made to act along it.*

We already have one arbitrary point O on the first one-vector. Take any other point P. The plane through P and the first one-vector meets the S_3 determined by the other two in a point Q. The first one-vector may now be replaced by two components, along OP and OQ. The other two one-vectors may be replaced by two one-vectors in their S_3, one of which passes through Q. Compounding the two one-vectors meeting at Q, we are left with three one-vectors, one of which acts along the arbitrary line OP.

THEOREM V. *c. Any arbitrary S_3 being chosen, two of the one-vectors can be made to act in it.*

By the above we may take any straight line in the S_3 as line of action of one of the one-vectors. The other two one-vectors lie in a second S_3 which will intersect the first in a line. One of the two one-vectors may then be taken to act along this line and the result follows.

THEOREM V. *d. The arbitrary straight line being chosen* (V. *b*) *the S_3 containing the other two one-vectors is determined, also the magnitude of the first one-vector and the system in the S_3.*

Suppose first, if possible, that a second reduction could lead to a

different S_3. Reverse the three one-vectors so found, and we have with the first three a system in equilibrium. The two S_3's have a common straight line, and one one-vector of each pair may be taken to act along it. We have now two one-vectors in one of the original S_3's, one one-vector along the arbitrary line and another one-vector. These latter two one-vectors may be replaced by two others in the S_3 determined by them, one of which may, as before, be taken in the original S_3 in which the other pair of one-vectors lies. We are left with three one-vectors in this S_3 and one outside it. This system can evidently not be in equilibrium unless the last one-vector vanishes. This involves, however, the one-vector along the arbitrary line meeting one of the two S_3's, the possibility of which has been already excluded.

The first part of the theorem is therefore proved by a *reductio ad absurdum*. A moment's consideration shows that two different reductions to the same straight line and S_3 are impossible, for it would lead to a one-vector along the straight line balancing a system in the S_3. Thus the whole theorem is proved.

THEOREM V.*e*. *The arbitrary S_3 being chosen* (V.*c*.), *the line of action of the third one-vector is determined, also the magnitude of the third one-vector and the system in S_3.*

The proof, being similar to the last, may be omitted.

THEOREM V.*f*. *If the line of action* (S_1) *of one of the one-vectors pass through an arbitrary point* (S_0), *the two other one-vectors lie in a fixed S_4 passing through the S_0.*

Let O be the point. Choose any other straight line through O. The plane through this line and the line of action of the original one-vector meets the conjugate S_3 of this latter line in a point P. The one-vector may now be replaced by one along the new line and one along OP. Make one of the pair of one-vectors in the S_3 pass through P, and we have reduced to a one-vector along the new line and two one-vectors, which, by construction, lie in the S_4 determined by O and the first S_3. Hence by (V.*d*.) the result follows.

THEOREM V.*g*. *If the S_3 of two of the one-vectors pass through an arbitrary plane* (S_2), *the remaining one-vector lies in another plane which intersects the first, and therefore lies with it in an S_4.*

Choose any other S_3 through the plane. The S_4 containing the two S_3's meets the third one-vector in a point O. The first two one-vectors may be replaced by one in the common plane, and a second which, by a known theorem in S_3, however we choose the first one-vector, passes

through a fixed point P of that plane. We may replace this second one-vector by a one-vector in the plane and one along OP. Compounding this with the one-vector already passing through P, we are left, outside the chosen S_3, with a one-vector in the plane determined by P and the original one-vector through O.

Hence, by V. *e.*, the theorem follows.

Similarly, the converse may be proved, and the theorem that, if the S_3 containing two of the one-vectors turn round a fixed point, the line of action of the third one-vector lies in a fixed S_4 passing through the point.

All these theorems for S_4 and S_5 will be found summed up in the following four theorems for the general case of S_n.

THEOREM VI. *Any system of one-vectors in "even space" S_n, where $n = 2m$, is reducible to $m\left(=\frac{n}{2}\right)$ one-vectors, lying in a covariant S_{n-1}.*

THEOREM VII. *a. Any system of one-vectors in "odd space" S_n, where $n = 2m-1$, is reducible to $m\left(=\frac{n+1}{2}\right)$ one-vectors, the line of action of one of which may be chosen arbitrarily. The S_{n-2} containing the remaining $(m-1)\left(=\frac{n-1}{2}\right)$ one-vectors is then determined, and so are the magnitude of the first one-vector and the system in the S_{n-2}.*

THEOREM VII. *b. Any S_{2k-1} in the S_{2m-1} being chosen, k of the one-vectors can be made to act in it. The $S_{2m-1-2k}$ containing the remaining $(m-k)$ one-vectors is then determined, and so are the systems in the S_{2k-1}, $S_{2m-1-2k}$ respectively.*

THEOREM VII. *c. Any S_{2p} in the S_{2m-1} being chosen, the S_{2k-1} containing k of the one-vectors can be made to pass through it, if $k > p$ (or to lie in it if $k < p$). The conjugate $S_{2m-1-2k}$ then lies in a fixed $S_{2m-2p-2}$ (or passes through a fixed $S_{2m-2p-2}$), intersecting the S_{2p} and therefore lying with it in an S_{2m-2}.*

Assuming these theorems to have been demonstrated for all values of n up to $(2m-2)$ inclusive, we can prove them generally by induction.

It is evident that, VII. being proved, VI. follows at once from Theorem III., Cor. III.

To prove Theorem VII. By Theorem III., Cor. III., the system may be replaced by a one-vector through an arbitrary point O, and a system in an S_{n-1} not passing through O. Since, therefor[illegible]

$n-1 = 2m-2$ the system is equivalent to m one-vectors, one of which passes through an arbitrary point O.

Choose any other point P. The plane through P and the one-vector through O meet the S_{n-1} of the other one-vectors in a point Q. Replace the one-vector through O by one-vectors along OP, OQ, and choose one of the remaining one-vectors to pass through Q. Compounding the two one-vectors through Q, the system is reduced to a one-vector along an arbitrary line OP and $(m-1)$ other one-vectors.

It is now obvious that, instead of a straight line, we may choose an S_3, and make two one-vectors act in it. Because, one one-vector being chosen in the S_3, the S_3 and the S_{n-2} containing the other $(m-2)$ one-vectors intersect in a straight line, which may be taken as line of action of one one-vector.

Similarly, an S_5, S_7, or any odd space may be arbitrarily chosen.

Next, to prove that the choice of one arbitrary line determines the conjugate S_{n-2}.

Suppose two reductions lead to two different S_{n-2}'s; reverse one system so obtained, and we have with the other a system in equilibrium. Choosing $(m-2)$ one-vectors of each system in the S_{n-4} common to both S_{n-2}'s, we have left three one-vectors in a perfectly determinate S_5; one of these three may, of course, be chosen in the S_{n-4}, since an S_5 and S_{n-4} have a common line. We are left with

$$\left.\begin{array}{c}\text{two one-vectors in an } S_3\\ \text{a system in an } S_{n-4}\end{array}\right\}.$$

and

In the general case, the S_3 and S_{n-4} have no common point, and therefore the whole system can only be in equilibrium if each partial system is in equilibrium by itself. This, however, leads at once to a special system in which the arbitrarily chosen line intersects its polar S_{n-2}, and hence, by a *reductio ad absurdum*, the theorem is proved.

It is now obvious that the magnitude of the one-vector along the arbitrary line and the system in the polar S_{n-2} are determined; otherwise we could obtain a system of one-vectors in an S_{n-2}, balanced by a single one-vector whose line of action lies entirely outside that S_{n-2}.

The proof of Theorems *b* and *c*, being very simple and identical in principle with the proofs of the corresponding theorems for five dimensions, may be omitted.

Counting up the Constants.—A system of one-vectors in S_n has $\frac{1}{2}n(n+1)$ constants. If we divide this up into an equivalent system in an S_k and one in an S_{n-k-1}, we have

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{2}k(k+1) - \tfrac{1}{2}(n-k-1)(n-k) = (k+1)(n-k)$$

constants still at our disposal; this is exactly the number of coordinates of an S_k, and we should therefore expect to be able to choose the S_k arbitrarily, as has been shown by the foregoing discussion to be the case.

THEOREM VIII. *When n is odd, and equal to* $2m-1$, *the m one-vectors to which a given (general) system can be reduced determines a fundamental* $(n+1)$*-pyramid of constant volume.*

We know that the theorem is true for space of three dimensions. We may therefore assume it to hold for S_{n-2}. It then follows for S_n. For, any arbitrary line being chosen, we have a determinate length along it representing the one-vector along it, and, corresponding to the other $(m-1)$ one-vectors, a fundamental $(n-1)$-pyramid of constant volume in a determinate S_{n-2} (polar S_{n-2}), not intersecting the line. We have, therefore, a fundamental $(n+1)$-pyramid whose volume remains constant, however we choose the $(n-1)$ one-vectors, as long as one line of action remains unaltered. Now take any other arbitrary line. Its polar S_{n-2} has an S_{n-4} common with the first S_{n-2}. Take any line in this S_{n-4}. We may arrange that a one-vector should act along this line in each reduction. It follows that the volume of the pyramid is the same for the two reductions.

COROLLARY. *When n is even, and equal to* $2m$, *the m one-vectors to which a given general system can be reduced determine an n-pyramid which is fundamental in the* S_{n-1} *containing it, and whose volume is constant.*

This last theorem puts us in a position to discuss the special cases in odd spaces. If the arbitrary line meets its polar S_{n-2}, the volume of the pyramid is zero, and the system can be reduced to one in space of $(n-2)$ dimensions, and therefore to $(m-1)$ one-vectors. Conversely, unless the volume of the pyramid is zero, the system cannot be reduced to $(m-1)$ one-vectors, or to a system in space of $(n-2)$ dimensions. Such a system we may call "once specialized." If the new fundamental $(n-1)$-pyramid collapse, we have a further reduction—a system twice specialized. In general, we may have a system specialized any number of times, up to $(m-1)$ times specialized, which would be a system equivalent to a single one-vector, or m times specialized, which would be a system in equilibrium.

THEOREM IX. *The necessary and sufficient conditions that a system should be equivalent to a single one-vector may be expressed in a single statement. The one-vectors along the edges of every tetrahedron of the fundamental* $(n+1)$*-pyramid must be equivalent to a single one-vector.*

That the conditions are necessary follows from the manner in

which we replace a one-vector by one-vectors along the fundamental $(n+1)$-pyramid. We have, therefore, only to show that the conditions are sufficient. Consider the case $n = 4$. Evidently, if the system be not equivalent to a single one-vector, when the one-vectors in every S_3 of the 5-pyramid are equivalent to a single one-vector, the two one-vectors to which the system is equivalent must be such that one passes through any arbitrary vertex of the pyramid; that is, every vertex of the fundamental pyramid lies in the characteristic S_3 defined by the two one-vectors, which is, of course, impossible.

When $n = 5$, by what has just been proved, if the one-vectors in every S_3 of the pyramid are equivalent to a single one-vector, this is true for every S_4 of the pyramid. Hence, if the whole system be not equivalent to a single one-vector, we may replace the system by two one-vectors—one through any vertex of the fundamental pyramid, the other in the opposite S_4. But the S_3 defined by these two one-vectors cannot pass through all the vertices of the fundamental pyramid. Proceeding in this way we may obviously extend the theorem to n dimensions.

Absolute Line Coordinates.—Let us choose definite directions along the edges of the fundamental $(n+1)$-pyramid as positive (viz., 12, 23, 31, 41, 42, 43, 51, 52, 53, 54, and so on). Now let us suppose a one-vector of unit magnitude to act in a definite sense along any straight line, and let us denote the ratio of the component one-vectors along any edge $\overline{rs}$ to the one-vector represented by that edge, attention being paid to sign, by p_{rs}. We have then, in the case when the system is equivalent to a single one-vector, a convenient system of absolute coordinates for the line of action including its sense. The construction of a line whose coordinates are thus defined follows by repeated application of the parallelogram law, and the same is true of the determination of the coordinates of a line whose position and sense are given. The identical relations satisfied by these $\frac{1}{2}(n+1)n$ coordinates, of which only $2(n-1)$ are independent, will be obtained by applying Theorem IX.

They are the well known equations*:—

$$p_{12}p_{34} + p_{23}p_{14} + p_{31}p_{24} = 0,$$
$$p_{12}p_{3r} + p_{23}p_{1r} + p_{31}p_{2r} = 0 \qquad (r = 4, 5, \ldots n),$$

* Each of these equations merely expresses the fact that the moment of the system constituted by the one-vectors acting along the edges of the corresponding tetrahedron about itself vanishes.

and similar equations for every four indices. There are evidently $\frac{1}{4!}(n+1)n(n-1)(n-2)$ such equations. Though necessary and sufficient, they are not all independent; there are syzygies between them. We have also a quadratic non-homogeneous equation, expressing the fact that the resultant of all the component one-vectors is of unit magnitude, viz.,

$$\Sigma a_{rr}^2 p_{rr}^2 + 2\Sigma a_{rr}a_{ij}\cos(rs, ij)p_{rr}p_{ij} = 1$$

with an obvious notation.

We may, of course, if we please, avoid this equation by using only the ratios of the p's.

It will be noted that this method of defining line coordinates is independent of point or plane coordinates; and the equations between them are obtained without any analysis, being immediate consequences of the definition.

An Essay towards the Generating Functions of Ternariants.

By Professor A. R. Forsyth. Received March 17th, 1898. Communicated April 7th, 1898.

The present paper contains an attempt to apply the principle of the method devised by Cayley for the construction of generating functions for binary forms to the corresponding problem for ternary forms. In the case of ternariants, regarded as determined by their leading coefficients, there is the difficulty that the universal concomitant u_x has unity for its leading coefficient, and that therefore no change is made in the leading coefficient of a concomitant on multiplying the concomitant by any power of u_x. It is necessary to take account of this consideration in discussing the problem from the side of the leading coefficients; accordingly, instead of dealing solely with the asyzygetic concomitants for the construction of a generating function, I have rejected those which are reducible by means of a power of u_x. Those which remain I have called a complete set of *hyposyzygetic* concomitants: in terms of u_x and of the members of such a set, every

concomitant, multiplied possibly by some power of u_x, can be expressed as a rational integral function.

The investigation aims at the construction of a numerical generating function for a quantic: as the concomitants considered for this purpose are hyposyzygetic, the numerical function is called sub-generating. The simple case of the ternary quadratic is completely solved, the finite form of the numerical sub-generating function being obtained. In the case of the ternary cubic, the expression in finite form of the sub-generating function has not been obtained; the expansion, however, is carried well beyond the limit of the degrees to which, as is known, the asyzygetic concomitants belong. By means of the sub-generating function, it appears that, among the asyzygetic concomitants, there are two syzygies (non-linear, of course) of degree 6: both of these are obtained.

Isobarism of Concomitants.

1. The bipartite ternary quantic of order n and class ν can be symbolically represented by $a_x^n u_a^\nu$, or by

$$(a_1x_1+a_2x_2+a_3x_3)^n\,(u_1a_1+u_2a_2+u_3a_3)^\nu,$$

where the coefficients a and a are umbral. To secure the isobarism of the quantic and to establish the isobarism of its concomitants, we assign* weights as follows:—

ρ to x_1 and a_1,	0 to u_1 and a_1,
1 to x_2 and a_2,	$\rho-1$ to u_2 and a_2,
0 to x_3 and a_3,	ρ to u_3 and a_3,

ρ being an arbitrary quantity. Using $a_{k,l,\kappa,\lambda}$ to denote the coefficient of

$$\frac{n!}{i!\,k!\,l!}\,\frac{\nu!}{\iota!\,\kappa!\,\lambda!}\,x_1^i x_2^k x_3^l u_1^\iota u_2^\kappa u_3^\lambda$$

in $a_x^n u_a^\nu$, where $i+k+l=n$ and $\iota+\kappa+\lambda=\nu$, we have

$$a_{k,l,\kappa,\lambda}=a_1^{n-k-l}a_2^k a_3^l a_1^{\nu-\kappa-\lambda}a_2^\kappa a_3^\lambda;$$

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* *American Journal of Mathematics*, Vol. XII. (1890), pp. 1–60, 115–160; in particular, p. 7. Other assignments, apparently more general, are possible; thus, we might assign weights σ, $\sigma+\rho-1$, $\sigma+\rho$ to u_1 and a_1, u_2 and a_2, u_3 and a_3, respectively; and weights τ, $\tau-\rho+1$, $\tau-\rho$ to a_1, a_2, a_3, respectively: where σ and τ are arbitrary. The effect is to add to the weight in the text a weight $n\sigma+\nu(\tau-\rho)$ to each coefficient; no discriminating advantage is gained.

Many of the statements made in the following investigation are proved in the paper just quoted.

consequently the weight of $a_{k,l,\kappa,\lambda}$ is

$$k(\rho-1)+l\rho+(\nu-\kappa-\lambda)\rho+\kappa$$
$$=\nu\rho+(k-\kappa)(\rho-1)+(l-\lambda)\rho.$$

Let there be a mixed concomitant Φ, of degree r in the coefficients, of order m in the point-variables x, and of class p in the line-variables u; and let its leading term, taken to be that which involves the highest power of x_1 and the highest power of u_1, be

$$(\Sigma N a^{r_1}_{k_1 l_1 \kappa_1 \lambda_1} a^{r_2}_{k_2 l_2 \kappa_2 \lambda_2} \dots a^{r_s}_{k_s l_s \kappa_s \lambda_s}) x_1^m u_1^p,$$

where N is numerical and changes from term to term in the coefficient of this leading term, and where

$$r_1+r_2+\dots+r_s=r.$$

It is known that, solely by means of direct differential operations, the concomitant is completely derivable from this leading term.

The weight of the leading term, say w_1, is

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(k_q-\kappa_q)(\rho-1)+(l_q-\lambda_q)\rho\right\}\right]+m\rho.$$

The concomitant Φ is covariantive for interchange of x_2 and x_3, which changes $a_{kl\kappa\lambda}$ into $a_{lk\lambda\kappa}$; consequently, the weight of the leading term is also

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(l_q-\lambda_q)(\rho-1)+(k_q-\kappa_q)\rho\right\}\right]+m\rho.$$

The concomitant is covariantive for interchange of x_1 and x_2, which changes $a_{k,l,\kappa,\lambda}$ into $a_{n-k-l,l,\nu-\kappa-\lambda,\lambda}$ and so changes the leading term into that which involves $x_2^m u_2^p$; consequently, as the concomitant is isobaric, the weight of this term is also w_1, and therefore

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(n-k_q-l_q-\nu+\kappa_q+\lambda_q)(\rho-1)+(l_q-\lambda_q)\rho\right\}\right]$$
$$+m+p(\rho-1).$$

Again, the concomitant is covariantive for interchange of x_1 and x_3; using the isobaric property as in the last case, we have

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(k_q-\kappa_q)(\rho-1)+(n-k_q-l_q-\nu+\kappa_q+\lambda_q)\rho\right\}\right]+p\rho.$$

Adding the last three expressions obtained for w_1, we have

$$3w_1=\Sigma\left[r_q\left\{3\nu\rho+(n-\nu)(2\rho-1)\right\}\right]+m(1+\rho)+p(2\rho-1)$$
$$=3m\rho+3r\left\{\nu\rho+\tfrac{1}{3}(n-\nu)(2\rho-1)\right\}-(m-p)(2\rho-1).$$

Hence the weight of the literal coefficient, say w, of the leading term is

$$w=w_1-m\rho$$
$$=r\nu\rho+\tfrac{1}{3}(2\rho-1)\left\{r(n-\nu)-m+p\right\}.$$

Moreover, equating the first two expressions given for w_1, we have

$$\Sigma \{r_q (k_q - \kappa_q)\} = \Sigma \{r_q (l_q - \lambda_q)\}.$$

Numerical Sub-Generating Functions for Unipartite Quantics.

2. Taking more particularly the case of unipartite ternary quantics of order n, represented symbolically by a_x^n, we write

$$a_x^n = \Sigma\Sigma \frac{n!}{i!\, k!\, l!} x_1^i x_2^k x_3^l a_{kl},$$

where $i+k+l=n$. Then the weight of a_{kl} is $k(\rho-1)+l\rho$. As before, let there be a concomitant Φ appertaining to a_x^n, of degree r in the coefficients, of order m in the point-variables x, and of class p in the line-variables u; and let its leading term be

$$(\Sigma N a_{k_1 l_1}^{r_1} a_{k_2 l_2}^{r_2} \dots a_{k_s l_s}^{r_s}) x_1^m u_1^p,$$

where N is numerical. Then the weight of the leading coefficient is

$$\tfrac{1}{3}(2\rho-1)(n\nu - m + p);$$

and also

$$\Sigma r_q (k_q - l_q) = 0.$$

Now, it is part of the general theory of ternariants that the leading coefficient of a concomitant—that is, the coefficient of the term which involves the highest power of x_1 and the highest power of u_1—satisfies the two partial differential equations

$$D_1 = 0, \quad D_6 = 0,$$

where

$$\left.\begin{aligned} D_1 &= \Sigma k a_{k-1,l+1} \frac{\partial a}{\partial a_{k,l}} \\ D_6 &= \Sigma l a_{k+1,l-1} \frac{\partial a}{\partial a_{k,l}} \end{aligned}\right\}.$$

These equations show that every such leading coefficient is a simultaneous invariant of the system of binary forms

$$\begin{aligned} &(a_{0,0}\mathfrak{x} *)^0, \\ &(a_{1,0}, a_{0,1}\mathfrak{x} *)^1, \\ &(a_{2,0}, a_{1,1}, a_{0,2}\mathfrak{x} *)^2, \\ &(a_{3,0}, a_{2,1}, a_{1,2}, a_{0,3}\mathfrak{x} *)^3, \\ &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots, \\ &(a_{n,0}, a_{n-1,1}, a_{n-2,2}, \dots, a_{2,n-2}, a_{1,n-1}, a_{0,n}\mathfrak{x} *)^n. \end{aligned}$$

Moreover, interchange of the binary variables (which need not be specified explicitly) changes a coefficient $a_{k,l}$ into $a_{l,k}$: hence it changes D_6 into D_1, and *vice versa*. We know from the theory of binariants that, instead of solving $D_1 = 0$ and $D_6 = 0$ as a simultaneous Jacobian system of partial differential equations, it is sufficient to take solutions of either of the equations, provided those solutions are of the weight proper to their degree; for, when they possess this appropriate weight, they satisfy the other of the equations.

3. Construct now a homogeneous function of degree r in the foregoing coefficients, say

$$\Sigma N_{1,2,\ldots} a_{k_1 l_1}^{r_1} a_{k_2 l_2}^{r_2} \ldots a_{k_s l_s}^{r_s},$$

where $r_1 + r_2 + \ldots + r_s = r$, retaining such combinations of degree r

as are of weight $\quad \frac{1}{3}(2\rho - 1)(nr - m + p)$,

say of weight $\quad \frac{1}{3}(2\rho - 1)(nr - a)$,

where a is some integer, not necessarily positive. Then we have

$$\frac{1}{3}(2\rho - 1)(nr - a) = \Sigma r_i \{k_i(\rho - 1) + l_i \rho\};$$

so that, as ρ is arbitrary,

$$\frac{2}{3}(nr - a) = \Sigma r_i (k_i + l_i),$$

$$\frac{1}{3}(nr - a) = \Sigma r_i k_i;$$

and therefore $\quad \Sigma r_i k_i = \Sigma r_i l_i,$

and $\quad \frac{1}{3}(2\rho - 1)(nr - a) = \Sigma r_i \{l_i(\rho - 1) + k_i \rho\},$

the last equation showing that, corresponding to any term, there exists in this general function the term obtained from it by interchanging the coefficients equidistant from the beginning and the end of any of the binary forms.

Let the function, thus constructed of the weight appropriate to the degree, be limited so as to satisfy one of the equations $D_1 = 0$, $D_6 = 0$; then, from what has been pointed out as regards binariants, it satisfies the other of the equations. Consequently, it appears that the imposition of the properties of homogeneity and isobarism upon the function enables us to consider only a single one of the two equations, the other of them (when that one is satisfied) being satisfied of itself.

For instance, in the case of the ternary cubic,

$$a_{00}(a_{30}a_{01}^2-2a_{21}a_{01}a_{10}+a_{12}a_{10}^2)$$

is a solution of $D_1=0$ but not of $D_6=0$; while

$$a_{30}a_{01}^3-3a_{21}a_{01}^2a_{10}+3a_{12}a_{01}a_{10}^2-a_{03}a_{10}^3$$

is a solution of $D_1=0$ and of $D_6=0$. Both functions are of degree 4; the weight of the former is

$$3(\rho-1)+2\rho=5\rho-3,$$

while the weight of the latter is

$$\begin{aligned}3(\rho-1)+3\rho &= 6\rho-3\\ &= \tfrac{1}{3}(2\rho-1)\{4.3-3\}.\end{aligned}$$

The latter is thus of a weight appropriate to the degree 4 in the coefficients of the ternary cubic; the former is not. In fact, the effective test is that *the weight should be divisible by* $2\rho-1$; the quotient, $3(nr-m+p)$, is known to be an integer because it is the index of the concomitant.

4. At this stage, it should be noticed that, so far as determinations are effected solely by the differential equations $D_1=0$ and $D_6=0$, the concomitant is determinate save as to a power of u_x, for the leading coefficient of a concomitant Θ is the same as that of Θu_x^i, where i is any integer. Further, so far as determinations are effected solely by the weight of the leading coefficient, it is the integer $m-p$ which is determined; as the association of any power of u_x with Θ, or the removal of any power of u_x from Θ, makes the same change in m and p, the integer $m-p$ is the same for Θ as for Θu_x^i. Consequently, if we limit ourselves to the two differential equations and to the consideration of the weight of the leading coefficient, the concomitant is determined save as to a power of u_x.

Suppose that a complete set of linearly independent solutions of the equations is known, that is, a set such that, in terms of its members, every other solution of the desired type can be expressed as a rational integral function. Each member of the set determines a concomitant, subject to a power of u_x; it will be assumed that such power is removed from the expression, so that the leading coefficient, taken solely in connexion with the two equations,* is associated with the

* The actual determination of m and p can be effected in other ways without ambiguity; in the present investigation, only the two equations $D_1=0$ and $D_6=0$ are being used.

simplest form of the concomitant. Now let any other solution of the equations of the desired type be given; it is expressible as a rational integral function of members of the preceding set, and therefore the corresponding concomitant in its simplest form is (save as to a possible power of u_x) expressible as a rational integral function of the members in the set of concomitants together with u_x. If the possible power of u_x indicated be u_x^0, the corresponding concomitant is reducible, and would, therefore, be excluded from a complete asyzygetic system of the quantic. If the possible power of u_x indicated be a positive power, the corresponding concomitant is irreducible in connexion with the set retained, but is reducible by a power of u_x; such a concomitant will be called *hyposyzygetic*. A system of concomitants of a quantic will be called a complete hyposyzygetic system of the quantic when every concomitant can be represented, except as to a power of u_x, as a rational integral function of u_x, and of the members of the system. In fact, if we use u_x as a reducing factor so that, if Θu_x^i is expressible in terms of other concomitants, we do not retain Θ, then an asyzygetic system would be reduced, by means of u_x, to a hyposyzygetic system. Manifestly, a hyposyzygetic system is intermediate between an algebraically complete system and a complete asyzygetic system.

In order, therefore, to obtain a complete hyposyzygetic system, it is necessary to obtain a complete system of linearly independent solutions of the equations $D_1 = 0$, $D_\theta = 0$.

5. Let $C_{n,r,a}$ be the number of different products that can be formed from the coefficients

$$a_{k,l} \quad \begin{pmatrix} k, l = 0, 1, \ldots, n \\ k+l \leqslant n \end{pmatrix},$$

which are of degree r in those coefficients, and are of weight $\frac{1}{3}(2\rho-1)(nr-a)$; and, associating an arbitrary constant with each of these products, let the sum of all these quantities be forced to satisfy one or other of the equations $D_1 = 0$, $D_\theta = 0$.

Consider, first, the effect of the operation D_θ upon the function thus constructed. It leaves the degree unaltered, and it decreases the weight by unity; so that, if $D_{n,r,a}$ is the number of different products that can be formed from the coefficients $a_{k,l}$ which are of degree r and of weight $\frac{1}{3}(2\rho-1)(nr-a)-1$, there must be $D_{n,r,a}$ relations among the arbitrary constants in order that the function may satisfy the equation $D_\theta = 0$. Hence there are $C_{n,r,a}-D_{n,r,a}$ linearly

independent solutions of $D_0 = 0$ which are of degree r and weight $\frac{1}{8}(2\rho-1)(nr-\alpha)$; and each of these is a solution of $D_1 = 0$.

Consider, secondly, the effect of the operation D_1 upon the function which has been constructed. It leaves the degree unaltered, and it increases the weight by unity; so that, if $E_{n,r,\alpha}$ is the number of different products that can be formed from the coefficients $a_{k,l}$ which are of degree r and of weight $\frac{1}{8}(2\rho-1)(nr-\alpha)+1$, there must be $E_{n,r,\alpha}$ relations among the arbitrary constants in order that the function may satisfy the equation $D_1 = 0$. Hence there are

$$C_{n,r,\alpha} - E_{n,r,\alpha}$$

linearly independent solutions of $D_1 = 0$ which are of degree r and weight $\frac{1}{8}(2\rho-1)(nr-\alpha)$; and each of these is a solution of $D_0 = 0$.

6. Since the weight of $a_{k,l}$ is

$$k(\rho-1)+l\rho,$$

the number of different products of degree r and weight w that can be formed from these quantities is the coefficient of $z^r x^w$ in the expansion of

$$\Pi\Pi \frac{1}{1-zx^{k(\rho-1)+l\rho}}$$

in ascending powers of z and x, the double product being taken for values of k and l, such that

$$k, l = 0, 1, \ldots, n,$$
$$k+l \leqslant n.$$

Let $$x^{\rho-1} = a, \quad x^{\rho} = b,$$

so that, as ρ is arbitrary, a and b can be regarded as replacing x and ρ; then

$$x^{\frac{1}{8}(2\rho-1)(nr-\alpha)} = (ab)^{\frac{1}{8}(nr-\alpha)},$$
$$x^{\frac{1}{8}(2\rho-1)(nr-\alpha)-1} = a^{\frac{1}{8}(nr-\alpha)+1} b^{\frac{1}{8}(nr-\alpha)-1},$$
$$x^{\frac{1}{8}(2\rho-1)(nr-\alpha)+1} = a^{\frac{1}{8}(nr-\alpha)-1} b^{\frac{1}{8}(nr-\alpha)+1}.$$

Hence $C_{n,r,\alpha}$ is the coefficient of $z^r a^{\frac{1}{8}(nr-\alpha)} b^{\frac{1}{8}(nr-\alpha)}$ in

$$\Pi\Pi \frac{1}{1-za^k b^l};$$

$D_{n,r,\alpha}$ is the coefficient of $z^r a^{\frac{1}{8}(nr-\alpha)+1} b^{\frac{1}{8}(nr-\alpha)-1}$ in the same expansion;

and $E_{n,r,a}$ is the coefficient of $z^r a^{\frac{1}{2}(nr-a)-1} b^{\frac{1}{2}(nr-a)+1}$ in the same expansion. Now

$$\begin{aligned}
&\Pi\Pi(1-za^kb^l)\\
&= (1-z)\\
&\quad\times(1-za)(1-zb)\\
&\quad\times(1-za^2)(1-zab)(1-zb^2)\\
&\quad\times \ldots\ldots\\
&\quad\vdots\\
&\quad\times(1-za^n)(1-za^{n-1}b)(1-za^{n-2}b^2)\ldots(1-za^2b^{n-2})(1-zab^{n-1})(1-zb^n),
\end{aligned}$$

which is symmetrical in a and b; consequently $D_{n,r,a} = E_{n,r,a}$, and the two numbers $C_{n,r,a} - D_{n,r,a}$ and $C_{n,r,a} - E_{n,r,a}$ are the same, as might have been expected from their significance. Denoting each of them by $N_{n,r,a}$, and taking

$$\theta_n = \tfrac{1}{3}(nr-a),$$

we have $N_{n,r,a}$, the number of linearly independent solutions of $D_1 = 0$ and $D_6 = 0$, as equal to

$$\text{coefficient of } z^r a^{\theta_n} b^{\theta_n} - \text{coefficient of } z^r a^{\theta_n+1} b^{\theta_n-1},$$

in the expansion of $\{\Pi\Pi(1-za^kb^l)\}^{-1}$,

in ascending powers of z, a, b.

7. That the solutions are linearly independent, so that $N_{n,r,a}$ is their exact number, follows as in the case of binariants; in fact, we are dealing with the two equations characteristic of invariants of a set of binary forms, and it is known* that the process gives the exact number.

Further, we know that each solution, common to $D_1 = 0$ and $D_6 = 0$, which is homogeneous and isobaric in the coefficients, determines a concomitant. Hence $N_{n,r,a}$ *is the number of hyposyzygetic concomitants appertaining to a ternary quantic of order n which are of degree r in the coefficients, and for which the order m, the class p, and*

* See, in particular, Elliott, "Some Properties of Homogeneous Isobaric Functions," *Proc. Lond. Math. Soc.*, Vol. XXIV. (1893), pp. 21–36.

by taking the terms next the end, beginning with the second line;

$$-az^3a^3b^3(1+z^3a^2b^2+z^6a^4b^4+z^9a^6b^6+\ldots),$$

by taking all the second terms, beginning with the third line; and so on. The sum therefore is

$$a(1-zab+z^2a^2b^2-z^3a^3b^3+\ldots)(1+z^3a^2b^2+z^6a^4b^4+z^9a^6b^6+\ldots)$$

$$=\frac{a}{(1+zab)(1-z^3a^2b^2)}.$$

Hence for our purpose V effectively is

$$\frac{1}{(1+zab)(1-z^3a^2b^2)};$$

and therefore the number of hyposyzygetic concomitants of degree r and order-class excess κ belonging to the ternary quadratic is

$$\text{co. } z^ra^\kappa b^\kappa \text{ in } \frac{1}{(1-z)(1-zab)(1+zab)(1-z^3a^2b^2)}$$

$$=\text{co. } z^ra^\kappa b^\kappa \text{ in } \frac{1}{1-z\,.\,1-z^2a^2b^2\,.\,1-z^3a^2b^2}.$$

The corresponding concomitants are therefore three. First, $r=1$, $\kappa=0$, $\alpha=m-p=2$; using the other differential equations of the concomitants, we find $p=0$, $m=2$; the concomitant is the ternary quadratic U. Secondly, $r=2$, $\kappa=2$, $\alpha=m-p=-2$; using the other differential equations, we find $m=0$, $p=2$; the concomitant is the reciprocant F. Thirdly, $r=3$, $\kappa=2$, $\alpha=0$; using the other differential equations, we find $m=0$, $p=0$; the concomitant is the discriminant Δ.

The complete hyposyzygetic system for the ternary quadratic is composed of U, F, Δ; with which must be associated the universal concomitant u_x. This system happens to be co-extensive with the asyzygetic system for the quadratic.

10. Another method of proceeding is as follows. The required number of concomitants is the excess of the coefficient of $z^ra^\kappa b^\kappa$ (say $P_{r,\kappa}$) over the coefficient of $z^ra^{\kappa+1}b^{\kappa-1}$ (say $Q_{r,\kappa}$) in the expansion of $\{(1-z)(1-za)(1-zb)(1-za^2)(1-zab)(1-zb^2)\}^{-1}$.

Then $P_{r,\kappa}$ is the number of distinct sets of solutions of the equations

$$\left.\begin{aligned} p_1+p_2+p_3+\ p_4+p_5+\ p_6 &= r \\ p_2\quad\ +2p_4+p_5\quad\ &= \kappa \\ p_3\quad\ \ +p_5+2p_6 &= \kappa \end{aligned}\right\};$$

and $Q_{r,\kappa}$ is the number of distinct sets of solutions of the equations

$$\left.\begin{aligned} q_1+q_2+q_3+\ q_4+q_5+\ q_6 &= r \\ q_3\quad +2q_4+q_5\quad\ \ &= \kappa+1 \\ q_3\quad\ \ +q_5+2q_6 &= \kappa-1 \end{aligned}\right\}.$$

Moreover, we have $2\kappa \leqslant nr$

in general, so that in this case we have

$$\kappa \leqslant r;$$

and $\kappa, = \frac{1}{3}(2r-\alpha)$, cannot be negative; so that in taking the coefficient of z^r, the values of κ to be considered are 0, 1, ..., r.

When $r=0$, $\kappa=0$; then $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is u_x.

When $r=1$, $\kappa=0$ or 1. When $\kappa=0$, then $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is U. When $\kappa=1$, then $P_{r,\kappa}=1$, $Q_{r,\kappa}=1$; there is no concomitant.

When $r=2$, $\kappa=0$, 1, 2. When $\kappa=0$, $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is U^2. When $\kappa=1$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant. When $\kappa=2$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=1$; the single concomitant is F.

When $r=3$, $\kappa=0$, 1, 2, 3. When $\kappa=0$, $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is U^3. When $\kappa=1$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant. When $\kappa=2$, $P_{r,\kappa}=4$, $Q_{r,\kappa}=2$; the two concomitants are UF, Δ. When $\kappa=3$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant.

When $r=4$, $\kappa=0$, 1, 2, 3, 4. When $\kappa=0$, $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is U^4. When $\kappa=1$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant. When $\kappa=2$, $P_{r,\kappa}=6$, $Q_{r,\kappa}=4$; the two concomitants are U^2F, $U\Delta$. When $\kappa=3$, $P_{r,\kappa}=6$, $Q_{r,\kappa}=6$; there is no concomitant. When $\kappa=4$, $P_{r,\kappa}=3$, $Q_{r,\kappa}=2$; the single concomitant is F^2.

It thus appears that, in the fourth degree, there is no new irreducible concomitant; and there is thus suggested the inference that the complete hyposyzygetic system of the ternary quadratic is made up of u_x, U, F, Δ.

11. The preceding methods, effective for the quadratic, do not appear to be effective for the cubic without considerable labour; the following method, however, is effective for the quadratic and promises to be possible for the cubic.

Let $ab = u^2$, $\frac{b}{a} = v^2$, so that $a = \frac{u}{v}$, $b = uv$. Then

$$V = \frac{1-v^2}{1-z\frac{u}{v}.1-zuv.1-z\frac{u^2}{v^2}.1-zu^2v^2}.$$

Now in V the terms required are of the form $z^t(ab)^\kappa$, that is, of the form $z^t u^{2\kappa}$; in other words, the aggregate of terms independent of v in the expansion of V is required. This aggregate manifestly is given by those terms of

$$\Sigma z^t u^{p+q+2r+2s} v^{-p+q-2r+2s} - \Sigma z^t u^{p+q+2r+2s} v^{2-p+q-2r+2s}$$

which are free from v; the index t being $p+q+r+s$. For appropriate terms from the first sum, we have

$$p+2r = q+2s,$$

$$t = p+q+r+s$$

$$= 2q-r+3s;$$

they lead to a sum Σ_1, where

$$\Sigma_1 = \Sigma\Sigma\Sigma z^{2q-r+3s} u^{2q+4s},$$

the summation extending over values of q and s, and of values of r consistent with

$$p+2r = q+2s.$$

The terms in the second set lead similarly to a sum Σ_2, where

$$\Sigma_2 = \Sigma\Sigma\Sigma z^{2q-r+3s+2} u^{2q+4s+2},$$

the summation extending over values of q and s, and of values of r consistent with

$$p+2r = q+2s+2.$$

The full aggregate of terms then is

$$\Sigma_1 - \Sigma_2.$$

The possible indices of u, of the forms $2q+4s$ and $2q+4s+2$, can be arranged in two sets, those evenly even and those oddly even.

Consider, first, those evenly even, in particular, the terms giving u^{4m}. In Σ_1, we therefore take

$$2q+4s = 4m,$$

so that

$$q = 2\theta, \quad s = m-\theta,$$

and θ ranges from 0 to m. Also

$$p+2r = q+2s = 2m,$$

so that r ranges from 0 to m; and

$$2q+3s = 3m+\theta.$$

Hence, in Σ_1, the coefficient of u^{4m} is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta-r}.$$

In Σ_2, we take $\quad 2q+4s+2 = 4m,$

so that $\quad q = 2\theta-1, \quad s = m-\theta,$

and θ ranges from 1 to m. Also

$$p+2r = q+2s+2 = 2m+1,$$

so that r ranges from 0 to m; and

$$2q+3s+2 = 3m+\theta.$$

Hence, in Σ_2, the coefficient of u^{4m} is

$$\sum_{\theta=1}^{m} \sum_{r=0}^{m} z^{3m+\theta-r}.$$

Consequently, in $\Sigma_1-\Sigma_2$, the coefficient of u^{4m} is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta-r} - \sum_{\theta=1}^{m} \sum_{r=0}^{m} z^{3m+\theta-r}$$

$$= \sum_{r=0}^{m} z^{3m-r}$$

$$= \sum_{\theta=0}^{m} z^{3m-\theta} = \sum_{\theta=0}^{m} z^{q+3s},$$

and $2q+4s = 4m$; so that the aggregate of the evenly even terms can be represented in the form

$$\Sigma (zu^2)^q (z^3u^4)^s,$$

where q is even for this aggregate.

Consider, secondly, the terms oddly even, in particular, the terms giving u^{4m+2}. In Σ_1, we therefore take

$$2q+4s = 4m+2,$$

so that $\quad q = 2\theta+1, \quad s = m-\theta,$

and θ ranges from 0 to m. Also

$$p+2r = q+2s = 2m+1,$$

so that r ranges from 0 to m; and

$$2q+3s = 3m+\theta+2.$$

Hence, in Σ_1, the coefficient of u^{4m+2} is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta+2-r}.$$

In Σ_2, we take $2q+4s+2 = 4m+2,$

so that $q = 2\theta, \quad s = m-\theta,$

and θ ranges from 0 to m. Also

$$p+2r = q+2s+2 = 2m+2,$$

so that r ranges from 0 to $m+1$; and

$$2q+3s+2 = 3m+\theta+2.$$

Hence, in Σ_2, the coefficient of u^{4m+2} is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m+1} z^{3m+\theta+2-r}.$$

Consequently, in $\Sigma_1 - \Sigma_2$, the coefficient of u^{4m+2} is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta+2-r} - \sum_{\theta=0}^{m} \sum_{r=0}^{m+1} z^{3m+\theta+2-r}$$

$$= -\sum_{\theta=0}^{m} z^{2m+\theta+1}$$

$$= -\sum_{\theta=0}^{m} z^{3m-\theta+1} = -\sum_{\theta=0}^{m} z^{q+3s},$$

and $2q+4s = 4m+2$; so that the aggregate of the oddly even terms can be represented in the form

$$-\Sigma\,(zu^2)^q\,(z^3u^4)^s,$$

where q is odd for this aggregate.

Hence the total aggregate of terms required is

$$\sum_{q \text{ even}} (zu^2)^q\,(z^3u^4)^s - \sum_{q \text{ odd}} (zu^2)^q\,(z^3u^4)^s$$

$$= \Sigma\,(-zu^2)^q\,(z^3u^4)^s,$$

for all values of q and s. The initial values for the even sum are $q = 0$, $s = 0$; and for the odd sum are $q = 1$, $s = 0$; hence this aggregate is

$$\sum_{q=0}^{\infty} (-zu^2)^q \sum_{s=0}^{\infty} (z^3u^4)^s$$

$$= \frac{1}{1+zu^2 \,.\, 1-z^3u^4},$$

and therefore the part of V required is

$$\frac{1}{1+zu^2 \,.\, 1-z^2u^4}$$

$$= \frac{1}{1+zab \,.\, 1-z^2a^2b^2}.$$

This result leads to the same sub-generating function for the quadratic as before.

Application to Ternary Cubic.

12. We now consider the unipartite ternary cubic. Here $n = 3$, so that, writing

$$\kappa = \tfrac{1}{3}(3r-m+p)$$

$$= r-\tfrac{1}{3}(m-p),$$

we require a number represented by

$$\text{coefficient of } z^r a^\kappa b^\kappa - \text{coefficient of } z^r a^{\kappa-1} b^{\kappa+1},$$

in the expansion of Δ^{-1}, where

$$\begin{aligned}\Delta = &(1-z)\\ &\times (1-za)(1-zb)\\ &\times (1-za^2)(1-zab)(1-zb^2)\\ &\times (1-za^3)(1-za^2b)(1-zab^2)(1-zb^3)\,;\end{aligned}$$

or, what is the same thing, the coefficient of $z^r a^\kappa b^\kappa$ in the expansion of

$$\frac{1-\dfrac{b}{a}}{\Delta}.$$

Write $\dfrac{b}{a} = v^2$, $ab = u^2$; then $z^r a^\kappa b^\kappa = z^r u^{2\kappa}$, and therefore what is required is the aggregate of terms independent of v in the expansion of

$$\frac{1}{1-z \,.\, 1-zu^2} V,$$

where

$$V = \frac{1-v^2}{1-z\frac{u}{v} \,.\, 1-zuv \,.\, 1-z\frac{u^2}{v^2} \,.\, 1-zu^2v^2 \,.\, 1-z\frac{u^3}{v^3} \,.\, 1-z\frac{u^3}{v} \,.\, 1-zu^3v \,.\, 1-zu^3v^3}.$$

This aggregate manifestly is given by the terms of

$$\Sigma z^t u^{\lambda+\mu+2(\pi+\rho)+3(\alpha+\beta+\gamma+\delta)} v^{-\lambda+\mu+2(-\pi+\rho)+(-3\alpha-\beta+\gamma+3\delta)}$$

$$-\Sigma z^t u^{\lambda+\mu+2(\pi+\rho)+3(\alpha+\beta+\gamma+\delta)} v^{2-\lambda+\mu+2(-\pi+\rho)+(-3\alpha-\beta+\gamma+3\delta)}$$

which are free from v; and the index of z is

$$t = \lambda+\mu+\pi+\rho+\alpha+\beta+\gamma+\delta.$$

For the appropriate terms of the first sum, we have

$$\lambda+2\pi+3\alpha+\beta = \mu+2\rho+3\delta+\gamma;$$

and then $$t = 2\mu+3\rho+2\gamma+4\delta-\pi-2\alpha = T,$$

say, and the index of u is

$$2(\mu+2\rho+\beta+2\gamma+3\delta) = 2\Theta,$$

say; so that the appropriate terms of the first sum are

$$\Sigma_1 = \Sigma z^T u^{2\Theta},$$

the summation extending over values of μ, ρ, β, γ, δ, and over values of λ, π, α, such that

$$\lambda+2\pi+3\alpha+2\beta+\gamma = \mu+2\rho+\beta+2\gamma+3\delta = \Theta,$$

that is, values of λ, π, α such that

$$\lambda+2\pi+3\alpha = \Theta-2\beta-\gamma.$$

The terms in the second set lead similarly to an aggregate

$$\Sigma_2 = \Sigma z^{T+2} u^{2\Theta+2},$$

with the same symbolic signification of T and Θ; the summation extends over values of μ, ρ, β, γ, δ, and over values of λ, π, α, such that

$$\lambda+2\pi+3\alpha = \Theta+2-2\beta-\gamma.$$

The full aggregate of the terms is consequently

$$\Sigma_1 z^T u^{2\Theta} - \Sigma_2 z^{T+2} u^{2\Theta+2},$$

where

$$\Theta = \mu+2\rho+\beta+2\gamma+3\delta,$$

$$T = 2\mu+3\rho+2\gamma+4\delta-\pi-2\alpha;$$

the summation in both cases extends over values of μ, ρ, β, γ, δ; and further over values of π and α which, in the first sum, are consistent with

$$\lambda+2\pi+3\alpha = \Theta-2\beta-\gamma,$$

and, in the second sum, are consistent with

$$\lambda+2\pi+3\alpha = \Theta+2-2\beta-\gamma.$$

13. It will be convenient to describe the terms in Σ_1 as the positive terms, and those in Σ_2 as the negative terms.

The positive terms that involve $u^{2\Phi}$ are given by values of the integers such that

$$\left.\begin{aligned} \mu+2\rho+3\delta &= \Phi- \beta-2\gamma \\ \lambda+2\pi+3\alpha &= \Phi-2\beta- \gamma \end{aligned}\right\};$$

so that sets of values of μ, ρ, δ; λ, π, α; are obtainable for particular values of β and γ. For each set of simultaneous values, the index of z is T, where

$$\begin{aligned} T &= 2\mu+3\rho+2\gamma+4\delta-\pi-2\alpha \\ &= \lambda+\mu+\pi+\rho+\alpha+\delta+\beta+\gamma. \end{aligned}$$

And the range of values of β and γ giving rise to possible terms is determined by the relations

$$\left.\begin{aligned} \beta+2\gamma &\leqslant \Phi \\ 2\beta+ \gamma &\leqslant \Phi \end{aligned}\right\}.$$

The negative terms that involve $u^{2\Phi}$ are given by values of the integers such that

$$\left.\begin{aligned} \mu'+2\rho'+3\delta' &= \Phi-1- \beta'-2\gamma' \\ \lambda'+2\pi'+3\alpha' &= \Phi+1-2\beta'- \gamma' \end{aligned}\right\};$$

the range of values of β' and γ' is determined by the relations

$$\left.\begin{aligned} \beta'+2\gamma' &\leqslant \Phi-1 \\ 2\beta'+ \gamma' &\leqslant \Phi+1 \end{aligned}\right\};$$

and the index of z for any set of simultaneous values is T', where

$$T' = \lambda'+\mu'+\pi'+\rho'+\alpha'+\delta'+\beta'+\gamma'.$$

Now we have

$$\left.\begin{aligned} (\beta'-1)+2(\gamma'+1) &\leqslant \Phi \\ 2(\beta'-1)+ (\gamma'+1) &\leqslant \Phi \end{aligned}\right\},$$

so that we may take, in general,

$$\beta'-1 = \beta, \quad \gamma'+1 = \gamma,$$

leading to

$$\beta'+\gamma' = \beta+\gamma;$$

and then

$$\begin{aligned} \mu'+2\rho'+3\delta' &= \Phi-(\beta'-1)-2(\gamma'+1) \\ &= \Phi-\beta-2\gamma, \\ \lambda'+2\pi'+3\alpha' &= \Phi-2(\beta'-1)-(\gamma'+1) \\ &= \Phi-2\beta-\gamma, \end{aligned}$$

so that, for all the cases in which the change from β' and γ' to β and γ is possible, we may take

$$\mu' = \mu, \quad \rho' = \rho, \quad \delta' = \delta,$$
$$\lambda' = \lambda, \quad \pi' = \pi, \quad \alpha' = \alpha.$$

In each such case, $$T' = T;$$

and, consequently, for each such case, the positive and the negative term cancel each other.

There therefore survive only such terms in which the change from β' and γ' to β and γ is not possible. Of these there are two sets, viz., those given by $\gamma = 0$ in the positive series, and those given by $\beta' = 0$ in the negative series.

When $\gamma = 0$, the range of values of β is from 0 to $[\frac{1}{2}\Phi]$, where $[m]$ denotes the positive integral part of m. For these, we have

$$\begin{aligned} T &= \lambda+\mu+\pi+\rho+\alpha+\delta+\beta \\ &= 2\Phi-2\beta-\pi-2\alpha-\rho-2\delta; \end{aligned}$$

the values of π, α, ρ, δ are such that

$$\left.\begin{aligned} 2\rho+3\delta &\leqslant \Phi-\beta \\ 2\pi+3\alpha &\leqslant \Phi-2\beta \end{aligned}\right\};$$

and thus the aggregate of the positive terms in the coefficient of $u^{2\Phi}$ is

$$\sum_{\beta=0}^{[\frac{1}{2}\Phi]} \sum_{\pi} \sum_{\alpha} \sum_{\rho} \sum_{\delta} z^T.$$

When $\beta' = 0$, the range of values of γ' is from 0 to $[\frac{1}{2}(\Phi-1)]$. For these we have

$$\begin{aligned} T' &= \lambda'+\mu+\pi'+\rho'+\alpha'+\delta'+\gamma' \\ &= 2\Phi-2\gamma'-\pi'-2\alpha'-\rho'-2\delta'; \end{aligned}$$

the values of π', α', ρ', δ' are such that

$$\left.\begin{aligned} 2\pi'+3\alpha' &\leqslant \Phi+1-\gamma' \\ 2\rho'+3\delta' &\leqslant \Phi-1-2\gamma' \end{aligned}\right\};$$

and thus the aggregate of the negative terms in the coefficient of $u^{2\Phi}$ is

$$-\sum_{\gamma'=0}^{[\frac{1}{2}(\Phi-1)]} \sum_{\pi'} \sum_{\alpha'} \sum_{\rho'} \sum_{\delta'} z^{T'}.$$

As the range of γ' is the same as that of β, except only for the value $\beta = \frac{1}{2}\Phi$ when Φ is an even integer, we may write $\gamma' = \beta$; for

the value $2\gamma' = \Phi$ admits of no solution of

$$2\rho' + 3\delta' \leqslant \Phi - 1 - 2\gamma',$$

and, consequently, no terms corresponding to this value are thus introduced into the negative aggregate. Also, for purposes of comparison, we write

$$\pi' = \rho_1, \quad \alpha' = \delta_1, \quad \rho' = \pi_1, \quad \delta' = \alpha_1;$$

so that the negative terms arise through values such that

$$\left.\begin{aligned} 2\rho_1 + 3\delta_1 &\leqslant \Phi + 1 - \beta \\ 2\pi_1 + 3\alpha_1 &\leqslant \Phi - 1 - 2\beta \end{aligned}\right\},$$

and T' is the same function of ρ_1, δ_1, π_1, α_1 as T is of ρ, δ, π, α.

Now the tableau of equations giving rise to positive terms is

$$\begin{aligned} 2\rho + 3\delta = \Phi, \Phi-1, \Phi-2, \Phi-3, \quad & 2\pi + 3\alpha = \Phi, \Phi-1, \Phi-2, \Phi-3, \Phi-4, \\ \Phi-1, \Phi-2, \Phi-3, \quad & \Phi-2, \Phi-3, \Phi-4, \\ \Phi-2, \Phi-3, \quad & \Phi-4, \\ \vdots \quad & \vdots \end{aligned}$$

and the corresponding tableau for negative terms is

$$\begin{aligned} 2\rho_1 + 3\delta_1 = \Phi+1, \Phi, \Phi-1, \Phi-2, \quad & 2\pi_1 + 3\alpha_1 = \Phi-1, \Phi-2, \Phi-3, \Phi-4, \Phi-5, \\ \Phi, \Phi-1, \Phi-2, \quad & \Phi-3, \Phi-4, \Phi-5, \\ \Phi-1, \Phi-2, \quad & \Phi-5, \\ \vdots \quad & \vdots \end{aligned}$$

When $2\rho + 3\delta$ and $2\rho_1 + 3\delta_1$ have the same value, and $2\pi + 3\alpha$ and $2\pi_1 + 3\alpha_1$ have the same value, the various forms of T are the same in number and value as the various forms of T'; and these corresponding terms cancel. When account is taken of all the sets that thus cancel, it appears that the surviving positive terms are given by

$$2\pi + 3\alpha = \Phi - 2\beta, \quad 2\rho + 3\delta \leqslant \Phi - \beta,$$

and the surviving negative terms by

$$2_1\pi + 3\alpha_1 \leqslant \Phi - 1 - 2\beta, \quad 2\rho_1 + 3\delta_1 = \Phi + 1 - \beta.$$

Some further removal of cancelling terms is possible, but the resulting conditions are more complicated, and do not prove so easy of application.

Accordingly, the required expansion of the function can be expressed in the form $\sum_{\Phi=0} u^{2\Phi}\{\Sigma z^T - \Sigma' z^{T'}\}$

where $T = 2\Phi - 2\beta - \pi - 2\alpha - \rho - 2\delta,$

the summation of z^T being for values of π, α, ρ, δ such that

$$2\pi + 3\alpha = \Phi - 2\beta, \quad 2\rho + 3\delta \leqslant \Phi - \beta;$$

and $T' = 2\Phi - 2\beta - \pi_1 - 2\alpha_1 - \rho_1 - 2\delta_1,$

the summation of $z^{T'}$ being for values of π_1, α_1, ρ_1, δ_1 such that

$$2\pi_1 + 3\alpha_1 \leqslant \Phi - 1 - 2\beta, \quad 2\rho_1 + 3\delta_1 = \Phi + 1 - \beta.$$

14. It has been established by Gordan* that the ternary cubic has 34 concomitants in its asyzygetic system, including itself and u_x; and that these concomitants are of various degrees from 0 (that of u_x) up to 12. It is thus of importance to possess the expansion of the preceding function, certainly complete as far as z^{12} inclusive; as a matter of fact, I have calculated it as far as z^{20} inclusive. Denoting this part by V_1, I find that the expression for V_1 is that given in Table I. (p. 509): the column down the page gives the powers of z, and the row across the page gives the powers of u; thus the coefficient of z^6 is

$$u^8 + 4u^{12} - u^{14} + 2u^{16},$$

and so for others.

The calculations proved rather long, but otherwise not difficult; they were effected as follows. It is easy to prove that, for any value of Φ, the lowest power of z that can occur in the coefficient of $u^{2\Phi}$ is $[\frac{1}{3}(2\Phi+2)]$; in order, therefore, to have the complete coefficient of z^{20}, it is sufficient to consider the values $\Phi = 0, 1, \ldots, 30$.

In the first place, a subsidiary table was formed of the values of $x+2y$ for values of x and y such that $2x+3y \leqslant m$, where m has the values 0, 1, ..., 31; and in each entry the values of $x+2y$ corresponding to $2x+3y = m$ were underlined. For instance, consider

$$2x + 3y \leqslant 9;$$

we have $y = 3;\ 2\ ;\ 1\ ;\ 0$

$$x = \underset{*}{0};\ 1, 0;\ \underset{*}{3}, 2, 1, 0;\ 4, 3, 2, 1, 0;$$

giving $x+2y = \underset{*}{6};\ 5, 4;\ \underset{*}{5}, 4, 3, 2;\ 4, 3, 2, 1, 0.$

* *Math. Ann.*, Bd. I. (1869), pp. 90–128.

TABLE I. (V_1).

	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60
0	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
1	...	−1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
2	...	...	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
3	...	...	1	0	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
4	...	...	...	...	2	−1	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
5	...	...	...	...	1	−1	3	−1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
6	...	...	...	...	1	0	4	−1	2	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
7	...	...	...	...	...	...	3	−1	4	0	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
8	...	...	...	...	...	...	2	0	5	0	4	−1	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
9	...	...	...	...	...	...	1	0	4	0	6	−1	4	−1	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
10	...	...	...	...	...	...	...	...	3	1	7	−1	8	−1	2	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:
11	...	...	...	...	...	...	...	...	2	1	6	0	10	−1	6	0	1	:	:	:	:	:	:	:	:	:	:	:	:	:	:
12	...	...	...	...	...	...	...	...	1	1	5	1	11	0	10	1	5	−1	1	:	:	:	:	:	:	:	:	:	:	:	:
13	...	...	...	...	...	...	...	...	...	...	3	1	9	0	13	1	9	−1	4	−1	:	:	:	:	:	:	:	:	:	:	:
14	...	...	...	...	...	...	...	...	...	...	2	1	8	2	13	2	14	0	10	−1	2	:	:	:	:	:	:	:	:	:	:
15	....	...	...	...	...	...	...	...	...	..	1	1	6	2	12	3	16	3	16	0	7	0	1	:	:	:	:	:	:	:	:
16	...	...	...	...	...	...	...	...	...	...	...	...	4	2	10	4	17	5	20	1	13	1	5	−1	1	:	:	:	:	:	:
17	...	...	...	...	...	...	...	...	...	...	...	...	2	1	8	4	15	8	22	2	19	3	11	−1	4	−1	:	:	:	:	:
18	...	...	...	...	...	...	...	...	...	...	...	...	1	1	6	4	14	9	23	4	24	5	21	1	11	−1	2	:	:	:	:
19	...	...	...	...	...	...	...	...	...	...	...	...	...	...	4	3	11	9	21	5	26	5	25	3	19	0	7	0	1	:	:
20	...	...	...	...	...	...	...	...	...	...	...	...	...	...	2	2	9	8	19	7	27	8	30	5	28	3	15	1	5	−1	1

The entry was made in the form

$$x+2y = \underline{6, 5^2}, 4^3, 3^2, 2^2, 1, 0.$$

When the values for $2x+3y \leqslant 10$

are required, they are the preceding set of values, together with those for which

$$2x+3y = 10,$$

and the result is $x+2y = \underline{6^2, 5^3}, 4^3, 3^2, 2^2, 1, 0.$

The values of Φ were then taken in succession; for each such value of Φ, positive terms are given by $\beta = 0$ to $[\frac{1}{2}\Phi]$, and negative terms by $\beta = 0$ to $[\frac{1}{2}(\Phi-1)]$. The values of $\pi+2\alpha$ and $\rho+2\delta$, such that

$$2\pi+3\alpha = \Phi-2\beta, \quad 2\rho+3\delta \leqslant \Phi-\beta,$$

for each of the values of β in succession, were extracted from the subsidiary table, and the corresponding values of T were obtained, all values greater than 20 being neglected. Similarly, for the negative terms. The quantity

$$\Sigma z^T - \Sigma z^{T'}$$

was then formed; this gave the expression for the coefficient of $u^{2\Phi}$. This was done for each of the values of Φ in succession, and the final result was arranged in powers of z, in the form given above.

15. The function which gives the number of hyposyzygetic concomitants of any assigned degree, and any assigned difference of order and class, by means of the coefficients of the appropriate power of z and power of u, is

$$\frac{V_1}{(1-z)(1-zu^2)}.$$

It is convenient to multiply numerator and denominator by $1+zu^2$; and, when the new form of the function is denoted by

$$\frac{N}{(1-z)(1-z^2u^4)},$$

the expression for N is given in Table II. (p. 511), the terms in z^1 and z^2 having disappeared. I have not, however, succeeded in reducing it to such terms that it will exhibit, in finite form, the expression for this sub-generating function of the ternary cubic. Various simplifications are possible, but they are not given, as they do not appear to be necessarily in the direct line leading to the function.

TABLE II. (N).

	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60
0	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3	...	...	1	1	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
4	...	...	...	1	2	0	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5	...	...	...	...	1	1	2	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	...	...	...	...	1	1	3	2	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	...	...	...	...	...	1	3	3	3	2	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	...	...	...	...	...	...	2	3	4	4	4	0	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9	...	...	...	...	...	...	1	2	4	5	6	3	3	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	...	...	...	...	...	...	...	1	3	5	7	5	7	3	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
11	...	...	...	...	...	...	...	...	2	4	7	7	9	7	5	2	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	...	...	...	...	...	...	...	...	1	3	6	7	11	10	9	7	5	0	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
13	...	...	...	...	...	...	...	...	...	1	4	6	10	11	13	11	10	4	3	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
14	...	...	...	...	...	...	...	...	...	...	2	4	9	11	13	15	15	9	9	3	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	...	...	...	...	...	...	...	...	...	...	1	3	7	10	14	16	18	17	16	10	6	2	1	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
16	...	...	...	...	...	...	...	...	...	...	...	1	5	8	12	16	20	21	23	17	13	8	5	0	1	⋮	⋮	⋮	⋮	⋮	⋮
17	...	...	...	...	...	...	...	...	...	...	...	...	2	5	10	14	19	25	27	22	20	16	12	4	3	⋮	⋮	⋮	⋮	⋮	⋮
18	...	...	...	...	...	...	...	...	...	...	...	...	1	3	7	12	18	24	31	26	26	24	24	12	10	3	1	⋮	⋮	⋮	⋮
19	...	...	...	...	...	...	...	...	...	...	...	...	...	1	5	9	15	23	30	28	30	29	30	24	20	11	6	2	1	⋮	⋮
20	...	...	...	...	...	...	...	...	...	...	...	...	...	...	2	6	12	19	28	28	32	34	35	30	31	22	15	8	5	0	1

16. As regards the 34 asyzygetic concomitants, their explicit expressions, when the cubic is taken in its canonical form, are given by Cayley.* Adopting his symbols, the combinations of the variables z and u corresponding to the respective concomitants are as follows:—

$$z = U;$$

$$z^3u^4 = \Theta;$$

$$z^3u^4 = H \;:\; z^3u^6 = B \;:\; z^3u^8 = P;$$

$$z^4u^6 = J \;:\; z^4u^8 = S, \Theta' : z^4u^{12} = F;$$

$$z^5u^8 = K \;:\; z^5u^{10} = B' \;:\; z^5u^{12} = Q, \overline{K};$$

$$z^6u^{10} = E \;:\; z^6u^{12} = T, \Theta'' : z^6u^{14} = \overline{E};$$

$$z^7u^{12} = K' \;:\; z^7u^{14} = B'' \;:\; z^7u^{16} = \overline{K}' : z^7u^{18} = \overline{M};$$

$$z^8u^{12} = \Psi \;:\; z^8u^{14} = E' \;:\; z^8u^{18} = \overline{J}, \overline{E}';$$

$$z^9u^{14} = M \;:\; z^9u^{18} = B''' \;:\; z^9u^{22} = \overline{M}';$$

$$z^{10}u^{18} = E'' : z^{10}u^{22} = \overline{E}'';$$

$$z^{11}u^{18} = M';$$

$$z^{12}u^{18} = \Omega \;:\; z^{12}u^{30} = \Pi;$$

unity corresponds to Λ, which represents $u_1x_1 + u_2x_2 + u_3x_3$. If, therefore, corresponding to each form $z^ru^{2\kappa}$, we construct an expression $1 - z^ru^{2\kappa}$, take the product of all these, and expand its reciprocal in powers of z and u, then the coefficient of $z^{r'}u^{2\kappa'}$ is the number of concomitants, say $f(r', k')$, determined by the numbers r' and k'. When full expansion of the sub-generating function is effected, let $f_1(r', k')$ be the coefficient of $z^{r'}u^{2\kappa'}$, which accordingly is the number of hyposyzygetic concomitants determined by the numbers r' and k'; consequently

$$f(r', k') - f_1(r', k')$$

is the number of syzygies among the concomitants, of degree r' and having their difference of order and class determined by k', which can be constructed out of the complete asyzygetic system of the cubic.

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* *American Journal of Mathematics*, Vol. IV. (1881), pp. 1–15; *Collected Mathematical Papers*, Vol. XI., pp. 342–356.

Retaining only terms involving powers of z not higher than the sixth, the preceding constructed function gives rise to an expression

$$\frac{L}{(1-z)(1-z^2u^4)},$$

where $L = (1+z^2u^4+z^6u^8+\ldots)(1+z^3u^6+z^6u^{12}+\ldots)(1+z^2u^8+z^6u^{16}+\ldots)$
$(1+z^4u^6+\ldots)(1+2z^4u^8+\ldots)(1+z^4u^{12}+\ldots)$
$(1+z^5u^8+\ldots)(1+z^5u^{10}+\ldots)(1+2z^5u^{12}+\ldots)$
$(1+z^6u^{10}+\ldots)(1+2z^6u^{12}+\ldots)(1+z^6u^{14}+\ldots)$

$$\begin{aligned}= 1&+z^2(u^4+u^6+u^8)\\ &+z^4(u^6+2u^8+u^{12})\\ &+z^5(u^8+u^{10}+2u^{12})\\ &+z^6(u^8+2u^{10}+4u^{12}+2u^{14}+u^{16})+\ldots.\end{aligned}$$

Now the sub-generating function is

$$\frac{N}{(1-z)(1-z^2u^4)},$$

where
$$\begin{aligned}N = 1&+z^2(u^4+u^6+u^8)\\ &+z^4(u^6+2u^8+u^{12})\\ &+z^5(u^8+u^{10}+2u^{12})\\ &+z^6(u^8+u^{10}+3u^{12}+2u^{14}+u^{16})+\ldots.\end{aligned}$$

We therefore infer that there is one syzygy among the concomitants determined by the type z^6u^{10}, and that there is one syzygy among those determined by the type z^6u^{12}.

17. To obtain the former of these syzygies, we construct the concomitants determined by the type z^6u^{10}; they are

E,	of class	2,	order	5,	
UB',	„	„	3,	„	6,
ΘJ,	„	„	3,	„	6,
HB,	„	„	3,	„	6,
$U\Theta B$,	„	„	5,	„	8;

and there is therefore a linear relation, with only numerical

coefficients, between

$$E\Lambda^3,\ UB'\Lambda^2,\ \Theta J\Lambda^2,\ HB\Lambda^2,\ U\Theta B.$$

This linear relation cannot involve $U\Theta B$; for every other term involves Λ^2, and the existence of such a relation would then imply the reducibility of Θ or B by the factor Λ, which is known not to be possible. The factor Λ^2 may therefore be removed; and the linear relation has the form

$$E\Lambda = aUB' + \beta\Theta J + \gamma HB,$$

where a, β, γ are numerical. The coefficients of the highest power of u_1, viz., of u_1^3, must be the same on the two sides; and therefore

$$\begin{aligned} x_1(abc+8l^3)(bx_2^3-cx_3^3)(2l^2x_1^2+bcx_2x_3) \\ = \quad & aU3l^2bc\,(cx_3^3-bx_2^3) \\ & +\beta\,(bcx_2x_3-l^2x_1^2)(abc+8l^3)\,x_1(bx-cx_3^3) \\ & +\gamma Hbc\,(cx_3^3-bx_2^3). \end{aligned}$$

In order that this may be identically satisfied, we must have

$$a=-1,\quad \beta=-2,\quad \gamma=3;$$

and therefore the syzygy, of degree 6, and excess of order over class equal to 3, is

$$E\Lambda + UB' + 2\Theta J = 3HB.$$

18. To obtain the other of the specified syzygies, we construct the concomitants determined by the type z^6u^{12}; they are

T,	of class	0,	order	0,
Θ'',	,, ,,	2,	,,	2,
UQ,	,, ,,	3,	,,	3,
$U\overline{K}$,	,, ,,	4,	,,	4,
ΘS,	,, ,,	2,	,,	2,
$\Theta\Theta'$,	,, ,,	4,	,,	4,
U^2F,	,, ,,	6,	,,	6,
HP,	,, ,,	3,	,,	3,
B^2,	,, ,,	6,	,,	6,
$U\Theta P$,	,, ,,	5,	,,	5,
Θ^3,	,, ,,	6,	,,	6;

and therefore the syzygy must have the form

$$\begin{aligned}\alpha\Theta^3+\beta B^2+\gamma U^2F &= \delta U\Theta P\Lambda\\ &\quad+(\epsilon U\bar{K}+Z\Theta\Theta')\,\Lambda^2\\ &\quad+(\eta UQ+\theta HP)\,\Lambda^3\\ &\quad+(\iota\Theta''+\kappa S\Theta)\,\Lambda^4+\lambda T\Lambda^6,\end{aligned}$$

where α, β, ..., λ are numerical.

The coefficients of the highest power of u_1, viz., of u_1^6, must be the same on the two sides. The coefficient on the right-hand side is manifestly divisible by x_1; and accordingly

$$\alpha\,(bcx_2x_3-l^2x_1^2)^3+\beta b^2c^2\,(cx_3^3-bx_2^3)^2+\gamma b^3c^3U^2,$$

which is the coefficient on the left-hand side, must be divisible by x_1. This requires the relations

$$\alpha=4\beta,\quad \gamma=-\beta;$$

and the coefficient on the left-hand side then is

$$\begin{aligned}&4\beta\,(bcx_2x_3-l^2x_1^2)^3+\beta b^2c^2\left\{(cx_3^3-bx_2^3)^2-U^2\right\}\\ &=-x_1\,12b^3c^3lx_2x_3\,(bx_2^3+cx_3^3)\,\beta\\ &\quad-x_1^2\,48b^3c^3l^2x_2^2x_3^2\beta\\ &\quad-x_1^3\,2ab^2c^2\,(bx_2^3+cx_3^3)\,\beta\\ &\quad+x_1^4\,12bcl\,(l^3-abc)\,x_2x_3\beta\\ &\quad-x_1^6\,(4l^6+a^2b^2c^2)\,\beta.\end{aligned}$$

Taking the coefficient of u_1^6 on the right-hand side, arranging it similarly in powers of x_1, and equating the terms that involve the respective powers of x_1, we have relations as follows.

From the terms involving x_1^1,

$$\delta bcx_2x_3\,(-lbc)(bx_2^3+cx_3^3)=-\,12b^3c^3lx_2x_3\,(bx_2^3+cx_3^3)\,\beta;$$

and therefore $$\delta=12\beta.$$

From the terms involving x_1^2,

$$\delta\,6lx_2x_3bcx_2x_3\,(-lbc)+\zeta 6b^2c^2l^2x_2^2x_3^2=-\,48b^3c^3l^2x_2^2x_3^2\beta,$$

so that $$-6\delta+6\zeta=-48\beta;$$

and therefore $$\zeta=4\beta.$$

From the terms involving x_1^3,

$$\begin{aligned}\delta l^3 bc\,(bx_2^3+cx_3^3)+\epsilon\,(abc+8l^3)\,bc\,(bx_2^3+cx_3^3)\\ +\eta\,(abc-10l^3)\,bc\,(bx_2^3+cx_3^3)-\theta lbc\,l^3(bx_2^3+cx_3^3)\\ =-2ab^3c^3\,(bx_2^3+cx_3^3)\,\beta,\end{aligned}$$

so that $\quad \delta l^3+\epsilon\,(abc+8l^3)+\eta\,(abc-10l^3)-\theta l^3=-2\beta abc,$

and therefore

$$\epsilon+\eta=-2\beta,$$
$$\delta+8\epsilon-10\eta-\theta=0,$$

that is,

$$\left.\begin{aligned}\epsilon+\eta&=-2\beta\\ \theta-8\epsilon+10\eta&=+12\beta\end{aligned}\right\}.$$

From the terms involving x_1^4,

$$\begin{aligned}\delta\,(-alb^2c^2+l^3bc\,6l)\,x_2x_3+\epsilon\,(abc+8l^3)\,bc\,6lx_2x_3\\ +\zeta\left\{lbc\,(abc+2l^3)-6bcl^4\right\}x_2x_3+\eta\,6lx_2x_3bc\,(abc-10l^3)\\ +\theta\,lbc\,(abc+2l^3)\,x_2x_3+\iota\,36bcl^4x_2x_3+\kappa Sbcx_2x_3\\ =\quad 12bcl\,(l^3-abc)\,x_2x_3\beta,\end{aligned}$$

so that, as $S=l\,(l^3-abc)$, we have

$$\begin{aligned}\delta\,(6l^3-abc)+6\epsilon\,(abc+8l^3)+\zeta\,(abc-4l^3)\\ +6\eta\,(abc-10l^3)+\theta\,(abc+2l^3)+36\iota l^3+\kappa\,(l^3-abc)\\ =12\beta\,(l^3-abc),\end{aligned}$$

and therefore

$$\begin{aligned}12\beta&=\quad 6\delta+48\epsilon-4\zeta-60\eta+2\theta+36\iota+\kappa,\\ -12\beta&=-\delta+\ 6\epsilon+\ \zeta+\ 6\eta+\ \theta-\kappa,\end{aligned}$$

which, by means of the preceding results, are easily reduced to

$$\theta-\kappa=8\beta,$$
$$36\iota+8\theta+\kappa=28\beta,$$

and, by combining these, also to the simpler form

$$\left.\begin{aligned}\theta+4\iota&=4\beta\\ \theta-\kappa\ &=8\beta\end{aligned}\right\}.$$

Lastly, from the terms involving x_1^6,

$$\begin{aligned}\delta l^3abc+\epsilon abc\,(abc+8l^3)-\zeta l^3\,(abc+2l^3)+\eta abc\,(abc-10l^3)-\theta abcl^3\\ -\iota\,(abc+2l^3)^2-\kappa l^2(l^4-abcl)+\lambda\,(a^2b^2c^2-20abcl^3-8l^6)\\ =-\beta\,(4l^6+a^2b^2c^2),\end{aligned}$$

and therefore

$$\left.\begin{aligned}-2\zeta-4\iota-\kappa-\ 8\lambda&=-4\beta\\ \delta+8\epsilon-\zeta-10\eta-\theta-4\iota+\kappa-20\lambda&=0\\ \epsilon+\eta-\iota+\lambda&=-\beta\end{aligned}\right\}.$$

Using the other results obtained, these give

$$\left.\begin{aligned} 4\beta+4\iota+\kappa &= -8\lambda \\ -4\beta-4\iota+\kappa &= 20\lambda \\ \beta+1 &= \lambda \end{aligned}\right\},$$

whence $$\lambda = 0, \quad \kappa = 0, \quad \beta+\iota = 0,$$

that is, $$\iota = -\beta.$$

Further, from the other relations, we have

$$\theta = 8\beta,$$
$$\epsilon = -\tfrac{4}{3}\beta,$$
$$\eta = -\tfrac{2}{3}\beta;$$

and therefore the syzygy, of degree 6 and of order equal to class, among the concomitants of the ternary cubic, is

$$4\Theta^3+B^3-U^2F = 12U\Theta P\Lambda+(4\Theta\Theta'-\tfrac{4}{3}U\overline{K})\Lambda^2+(8HP-\tfrac{2}{3}UQ)\Lambda^3-\Theta''\Lambda^4.$$

The two syzygies thus obtained are of degree 6; all others are of higher degree, and the simplest of them can be obtained in a similar manner. But, before the aggregate of syzygies can be obtained, it seems necessary to possess the expression for the sub-generating function in finite form.

Note on Bipartite Quantics.

19. A similar investigation applies to the biternary quantic of order n in the point-variables and class ν in the line-variables, and the result is to show that the number of hyposyzygetic concomitants of degree r, with their order and class connected by the relation

$$m-p = \alpha,$$

is equal to

$$\text{coefficient of } z^r a^{\phi_n} b^{\phi_n} - \text{coefficient of } z^r a^{\phi_n+1} b^{\phi_n-1}$$

in the expansion of $$\frac{1}{\Pi\Pi\Pi\Pi\,(1-za^{k-\kappa}b^{l-\lambda})},$$

the product being taken over the terms corresponding to

$$k, l = 0, 1, \ldots, n, \quad k+l \leqslant n,$$
$$\kappa, \lambda = 0, 1, \ldots, \nu, \quad \kappa+\lambda \leqslant \nu;$$

and the number ϕ_n is given by

$$\phi_n = \tfrac{1}{3}\{(n-\nu)\,r-m+p\}.$$

Thursday, May 12th, 1898.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Thirteen members present and a visitor.

The provisional action of the officers and members (as noted under April 7th) was ratified.

Miss M. E. Barwell and Mr. W. F. Sheppard were admitted into the Society.

Messrs. William Edward Philip, B.A., Fellow of Clare College, Cambridge; Thomas John I'Anson Bromwich, B.A., Fellow of St. John's College, Cambridge; and John Gaston Leathem, M.A., Fellow of St. John's College, Cambridge, and Isaac Newton Student, were elected members.

Mr. Love communicated a few Notes on some fundamental properties of Manifolds. Mr. Macaulay joined in a discussion of the communication.

The President read abstracts of the following papers:—"On the Numerical Value of $\int_0^h e^{x^2} dx$," by Mr. H. G. Dawson; and by Prof. Lamb, "On the Reflection and Transmission of Electric Waves by a Metallic Grating."

Lt.-Col. Cunningham gave an account of "The 77-Squares Puzzle" constructed by Major W. H. Turton, R.E., and exhibited a cardboard specimen of the puzzle. A vote of thanks was unanimously passed to Major Turton for the communication and for the present of the puzzle.* Lt.-Col. Cunningham also communicated the discovery of several high primes of 9, 10, and 11 figures.

Impromptu communications were made by the President—a remark in connexion with the figure of Pascal's theorem—and by Mr. Hargreaves.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LXII., Nos. 388–392.

"Mathematical Questions with their Solutions from the 'Educational Times,'" Vol. LXVIII.

"Journal of the Institute of Actuaries," Vol. XXXIV., Pt. 1, No. 189; April, 1898.

"Mitteilungen der Mathematischen Gesellschaft in Hamburg, Bd. III., Heft 7, 1897, Heft 8, 1898; Leipzig.

"Nyt Tidsskrift for Matematik," A, Aarg. 8, Nr. 7–8; B, Aarg. 9, Nr. 1, 1898; Copenhagen, 1897.

* For a description see Ball's *Mathematical Recreations and Problems* (Third Edition).

"Monatshefte für Mathematik und Physik," Jahrg. IX., Pt. 2; Vienna, 1898.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Série 2, Tome I., Livr. 4–5; La Haye, 1898.

"Bulletin of the American Mathematical Society," Series 2, Vol. IV., No. 7; New York, April, 1898.

"Science Abstracts," Vol. I., Pt. 3; London, March, 1898.

"Bulletin des Sciences Mathématiques," Tome XXII., Mars, Av.; Paris, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. IV., Fasc. 3–4; Napoli, 1898.

"Bulletin de la Société Mathématique de France," Tome XXVI., Nos. 1, 2; Paris, 1898.

"The John Crerar Library, Third Annual Report, for 1897"; Chicago, 1898.

"Annali di Matematica," Serie 3, Tomo I., Fasc. 1; Milano, 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 6, 7, 8; Roma, 1898.

"Educational Times," May, 1898.

"Transactions of the Cambridge Philosophical Society," Vol. XVI., Pt. 4; 1898.

"Indian Engineering," Vol. XXIII., Nos. 11–16, March 12–April 16, 1898.

"Transactions of the Royal Irish Academy," Vol. XXXI., Pts. 1–6; Dublin, Nov., 1896–March, 1898.

"International Catalogue of Scientific Literature." Report of the Committee of the Royal Society of London, with Schedules of Classification.

"The 77-Squares Puzzle," presented by the Author, Major W. H. Turton, R.E.

On the Numerical Value of $\int_0^h e^{x^2} dx$. *By* H. G. Dawson. Received April 30th, 1898. Communicated May 12th, 1898.

Starting from the two known equations

$$e^{-x^2} + e^{-(x-a)^2} + e^{-(x+a)^2} + e^{-(x-2a)^2} + e^{-(x+2a)^2} + \&c.$$
$$= \frac{\sqrt{\pi}}{a}\left(1 + 2e^{-(\pi^2/a^2)} \cos\frac{2\pi x}{a} + 2e^{-(4\pi^2/a^2)} \cos\frac{4\pi x}{a} + \&c.\right),$$

$$e^{-x^2} - e^{-(x-a)^2} - e^{-(x+a)^2} + e^{-(x-2a)^2} + e^{-(x+2a)^2} - \&c.$$
$$= \frac{2\sqrt{\pi}}{a}\left(e^{-(\pi^2/a^2)} \cos\frac{\pi x}{a} + e^{-(9\pi^2/a^2)} \cos\frac{3\pi x}{a} + \&c.\right),$$

we can, by putting $a = 1$, and making a few obvious transformations, deduce the two equations

$$e^{-x^2}\phi(x) = \sqrt{\pi}\left(1 + \frac{2\cos 2\pi x}{e^{\pi^2}} + \frac{2\cos 4\pi x}{e^{4\pi^2}} + \frac{2\cos 6\pi x}{e^{9\pi^2}} + \&c.\right),$$

$$e^{-x^2}\psi(x) = \sqrt{\pi}\, e^{\frac{1}{4}}\left(1 - \frac{2\cos 2\pi x}{e^{\pi^2}} + \frac{2\cos 4\pi x}{e^{4\pi^2}} - \frac{2\cos 6\pi x}{e^{9\pi^2}} + \&c.\right),$$

where $\phi(x) \equiv 1 + \frac{e^{2x}+e^{-2x}}{e} + \frac{e^{4x}+e^{-4x}}{e^4} + \ldots + \frac{e^{2mx}+e^{-2mx}}{e^{m^2}} + \ldots,$

and $\psi(x) = e^x + e^{-x} + \frac{e^{3x}+e^{-3x}}{e^3} + \ldots + \frac{e^{(2m+1)x}+e^{-(2m+1)x}}{e^{m(m+1)}} + \ldots.$

Approximately we have

$$\sqrt{\pi}\, e^{x^2}\left(1 + \frac{2\cos 2\pi x}{e^{\pi^2}}\right) = \phi_1(x),$$

$$e^{\frac{1}{4}}\sqrt{\pi}\, e^{x^2}\left(1 - \frac{2\cos 2\pi x}{e^{\pi^2}}\right) = \psi_1(x);$$

whence $2e^{\frac{1}{4}}\sqrt{\pi}\int_0^h e^{x^2}dx = \int_0^h \left\{e^{\frac{1}{4}}\phi_1(x) + \psi_1(x)\right\} dx \quad (h>2);$

therefore $2\sqrt{\pi}\int_0^h e^{x^2}dx = \int_0^h \left\{\phi_1(x) + e^{-\frac{1}{4}}\psi_1(x)\right\} dx,$

$$2\sqrt{\pi}\int_0^h e^{x^2}dx = \int_0^h \phi_1(x)\,dx + e^{-\frac{1}{4}}\int_0^h \psi_1(x)\,dx.$$

Now $\int_0^h \phi_1(x)\,dx = h + \frac{e^{2h}-e^{-2h}}{2e} + \frac{e^{4h}-e^{-4h}}{4e^4} + \frac{e^{6h}-e^{-6h}}{6e^9} + \ldots \equiv \psi_2(h),$

$$\int_0^h \psi_1(x)\,dx = \frac{e^h - e^{-h}}{1} + \frac{e^{3h}-e^{-3h}}{3e^2} + \frac{e^{5h}-e^{-5h}}{5e^6} + \ldots \equiv \chi_2(h).$$

Now, by aid of the tables given by Professor Newman in *Cambridge Philosophical Transactions*, Vol. XIII., supplemented by those of Dr. Glaisher, which are printed immediately after Professor Newman's, it was found, in general, possible to write down the values of $\phi_2(h)$, $\chi_2(h)$.

When it became necessary to calculate positive powers of e whose index was greater than 2, Dr. Glaisher's Table II. was no longer available, and then use was made of Guderman's *Theorie der Potenzial Functionen*, Tafel II., in conjunction with Vega's *Thesaurus Logarithmorum* (Leipzig, 1794).

The original terms of the series $\phi_2(h)$, $\chi_2(h)$ were taken to eight places of decimals.

By this means the values of $2\sqrt{\pi}\int_0^h e^{x^2}dx$ were found to eight [illegible]or values (of h) which increased from 0 to 2 by intervals of [illegible]ade ·01.

The values of $\int_0^h e^{x^2}dx$ were calculated by aid of an ordinary seven-place set of logarithm tables.

In this calculation we have neglected $\int_0^h e^{x^2}\frac{\cos 4\pi x}{e^{4\pi^2}}dx$ and smaller quantities; the value of

$$\int_0^h e^{x^2}\frac{\cos 4\pi x}{e^{4\pi^2}}dx < \frac{1}{4\pi}e^{-4(\pi^2-1)} < \frac{1}{12}e^{-34} < 2\times 10^{-16}.$$

I append tables of the values of $\int_0^h e^{x^2}dx$.

h	$\int_0^h e^{x^2}dx$	h	$\int_0^h e^{x^2}dx$	h	$\int_0^h e^{x^2}dx$
·01	·010000	·37	·387601	·73	·883332
·02	·020003	·38	·399111	·74	·900496
·03	·030008	·39	·410709	·75	·917916
·04	·040021	·40	·422397	·76	·935600
·05	·050042	·41	·434180	·77	·953553
·06	·060073	·42	·446060	·78	·971785
·07	·070114	·43	·458040	·79	·990305
·08	·080171	·44	·470122	·80	1·009120
·09	·090244	·45	·482313	·81	1·028239
·10	·100334	·46	·494614	·82	1·047669
·11	·110445	·47	·507027	·83	1·067419
·12	·120578	·48	·519558	·84	1·087500
·13	·130736	·49	·532209	·85	1·107922
·14	·140920	·50	·544987	·86	1·128696
·15	·151132	·51	·557892	·87	1·149828
·16	·161376	·52	·570930	·88	1·171331
·17	·171652	·53	·584104	·89	1·193216
·18	·181963	·54	·597418	·90	1·215497
·19	·192311	·55	·610876	·91	1·238180
·20	·202699	·56	·624484	·92	1·261280
·21	·213128	·57	·638244	·93	1·284809
·22	·223602	·58	·652163	·94	1·308782
·23	·234121	·59	·666243	·95	1·333207
·24	·244689	·60	·680492	·96	1·358100
·25	·255307	·61	·694912	·97	1·383476
·26	·265979	·62	·709509	·98	1·409349
·27	·276707	·63	·724289	·99	1·435735
·28	·287493	·64	·739255	1·00	1·462652
·29	·298339	·65	·754415	1·01	1·490109
·30	·309248	·66	·769772	1·02	1·518127
·31	·320223	·67	·785335	1·03	1·546721
·32	·331266	·68	·801106	1·04	1·575912
·33	·342381	·69	·817094	1·05	1·605716
·34	·353568	·70	·833303	1·06	1·636153
·35	·364833	·71	·849742	1·07	1·667240
·36	·376176	·72	·866415	1·08	1·699002

h	$\int_0^h e^{x^2}dx$	h	$\int_0^h e^{x^2}dx$	h	$\int_0^h e^{x^2}dx$
1·09	1·731457	1·40	3·240894	1·71	6·886604
1·10	1·764627	1·41	3·312892	1·72	7·076003
1·11	1·798534	1·42	3·386950	1·73	7·272034
1·12	1·833201	1·43	3·463141	1·74	7·474966
1·13	1·868655	1·44	3·541542	1·75	7·685085
1·14	1·904921	1·45	3·622235	1·76	7·902688
1·15	1·942021	1·46	3·705302	1·77	8·128087
1·16	1·979985	1·47	3·790831	1·78	8·361608
1·17	2·018840	1·48	3·878912	1·79	8·603593
1·18	2·058614	1·49	3·969639	1·80	8·854398
1·19	2·099341	1·50	4·063110	1·81	9·114396
1·20	2·141046	1·51	4·159428	1·82	9·383977
1·21	2·183766	1·52	4·258700	1·83	9·663552
1·22	2·227532	1·53	4·361035	1·84	9·953550
1·23	2·272378	1·54	4·466551	1·85	10·254406
1·24	2·318340	1·55	4·575367	1·86	10·566612
1·25	2·365457	1·56	4·687609	1·87	10·890680
1·26	2·413770	1·57	4·803408	1·88	11·227050
1·27	2·463311	1·58	4·922901	1·89	11·576332
1·28	2·514128	1·59	5·046230	1·90	11·939070
1·29	2·566264	1·60	5·173545	1·91	12·315857
1 30	2·619762	1·61	5·304999	1·92	12·703175
1·31	2·674669	1·62	5·440755	1·93	13·114102
1·32	2·731034	1·63	5·580982	1·94	13·536895
1·33	2·788907	1·64	5·725857	1 95	13·976416
1·34	2·848341	1·65	5·875565	1·96	14·433417
1·35	2·909391	1·66	6·030289	1 97	14·908699
1·36	2 972109	1·67	6·190240	1·98	15·403060
1·37	3·036555	1·68	6 355621	1·99	15·917402
1·38	3·102793	1·69	6·526656	2·00	16·45263
1·39	3·170884	1·70	6·703571		

The foregoing table is true to six figures; two tests can be used to try its accuracy; the first is as follows:—if u_h denote $\int_0^h e^{x^2}dx$, then

$$\Delta^2 u_h = 2h\delta\,(1+h\delta)\,\Delta' u_h.$$

The second is $\Delta u_h = e^{h^2}\left\{1+h\delta+\frac{1}{3}\,(1+2h^2)\,\delta^2\right\}\delta,$

where $\delta = \cdot 01$, $\Delta u_h = u_{(h+\delta)} - u_h$, $\Delta' u_h = u_h - u_{(h-\delta)}$.

On the Reflection and Transmission of Electric Waves by a Metallic Grating.

By Prof. HORACE LAMB, F.R.S. Received May 11th, 1898. Read May 12th, 1898.

The main problem of this paper consists in the calculation of the disturbance produced in a train of electric waves by a plane grating composed of parallel, equal, and equidistant metallic strips. The treatment is approximate, and involves the assumption that the wave-length is large compared with the distance between the centres of consecutive strips; the application is, therefore, rather to Hertzian waves than to phenomena of ordinary Optics. The previous mathematical literature bearing directly on the subject consists of an investigation by Prof. J. J. Thomson,* and of two papers by Lord Rayleigh.† It will be seen that the author is under great obligations to each of these sources.

The work of §§ 1, 2, 3 is preliminary, and consists in the solution of several electrostatical problems. With modern methods these present no difficulty; but the results appeared to possess sufficient interest of their own to warrant the expenditure of a little trouble in the way of graphical representation. The diagrams obtained serve, moreover, to illustrate very clearly some of the conclusions arrived at afterwards in the discussion of the wave problem.

The problem of § 4 is introduced on account of its close connexion with what precedes, although it has no bearing on the main subject of the paper.

1. Let it be required to find the disturbance produced in a uniform electric field by a grating (of the kind above described) placed with its plane perpendicular to the lines of force. We will suppose that the axis of x is normal to the plane of the grating, and the axis of y in this plane, at right angles to the lengths of the strips.

In the case of a field symmetrical on the two sides of the grating, it will be found that (with a special choice of units) the potential- and stream-functions are given by the formula

$$\cosh w = \mu \cosh z, \qquad (1)‡$$

* *Recent Researches on Electricity and Magnetism*, § 359. The problem attacked is that of reflection by a grating of cylindrical wires. Some notes on this question are included in the present paper, § 7.

† *Phil. Mag.*, April and July, 1897. See also *Theory of Sound*, § 292.

‡ [*July*, 1898.—This is not new, as the author had supposed. It was set in the Mathematical Tripos, Part II., 1895, and is due to Mr. Larmor.]

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where $$w = \phi + i\psi, \quad z = x + iy, \tag{2}$$

and the constant μ is supposed greater than unity. This result was obtained originally by a modification of Schwarz's method, but it will be sufficient here to verify it *à posteriori*. The formula (1) gives

$$\left.\begin{aligned} \cosh\phi\cos\psi &= \mu\cosh x\cos y \\ \sinh\phi\sin\psi &= \mu\sinh x\sin y \end{aligned}\right\}. \tag{3}$$

The locus $\phi = 0$ consists of those portions of the axis of y for which

$$1 > \mu\cos y > -1;$$

these represent the breadths of the metallic strips, so that (on the scale of our formulæ) the half-breadth of a strip is $\sin^{-1}(1/\mu)$. For other portions of the plane, ϕ is, so far, indeterminate as to sign; we will take it to be everywhere positive. Then ψ is indeterminate to the extent of a term $2s\pi$, but we may suppose that the lines $\psi = 0$, $\psi = \pm\pi$, $\psi = \pm 2\pi$, ... consist partly of the lines $y = 0$, $y = \pm\pi$, $y = \pm 2\pi$, ..., respectively, and partly of those portions of the axis of y for which

$$|\ \mu\cos y\ | > 1;$$

these correspond to the widths of the apertures between the successive strips. The half-breadth of an aperture, on the present scale, is $\cos^{-1}(1/\mu)$. At the centres of the apertures—for example, at the origin—we have $\phi = \cosh^{-1}\mu$; these are points of zero force, and the equipotential- and force-lines there meet at angles of 45°.

I have thought it worth while to trace the curves

$$\phi = \text{const.}, \quad \psi = \text{const.},$$

using the formula

$$\left.\begin{matrix} \cosh 2x \\ \cos 2y \end{matrix}\right\} = p \pm \sqrt{(p^2 - 2q + 1)}, \tag{4}$$

where

$$p = \frac{1}{2\mu^2}(\cosh 2\phi + \cos 2\psi), \quad q = \frac{1}{2\mu^2}(1 + \cosh 2\phi\cos 2\psi); \tag{5}$$

this is easily deduced from (3). The value of μ adopted for convenience of calculation was

$$\mu = \cosh\tfrac{1}{5}\pi = 1{\cdot}2040,$$

whence $$\sin^{-1}(1/\mu) = {\cdot}312\pi, \quad \cos^{-1}(1/\mu) = {\cdot}188\pi;$$

this gives the relative breadths of the strips and apertures. The curves are shewn in Fig. 1.

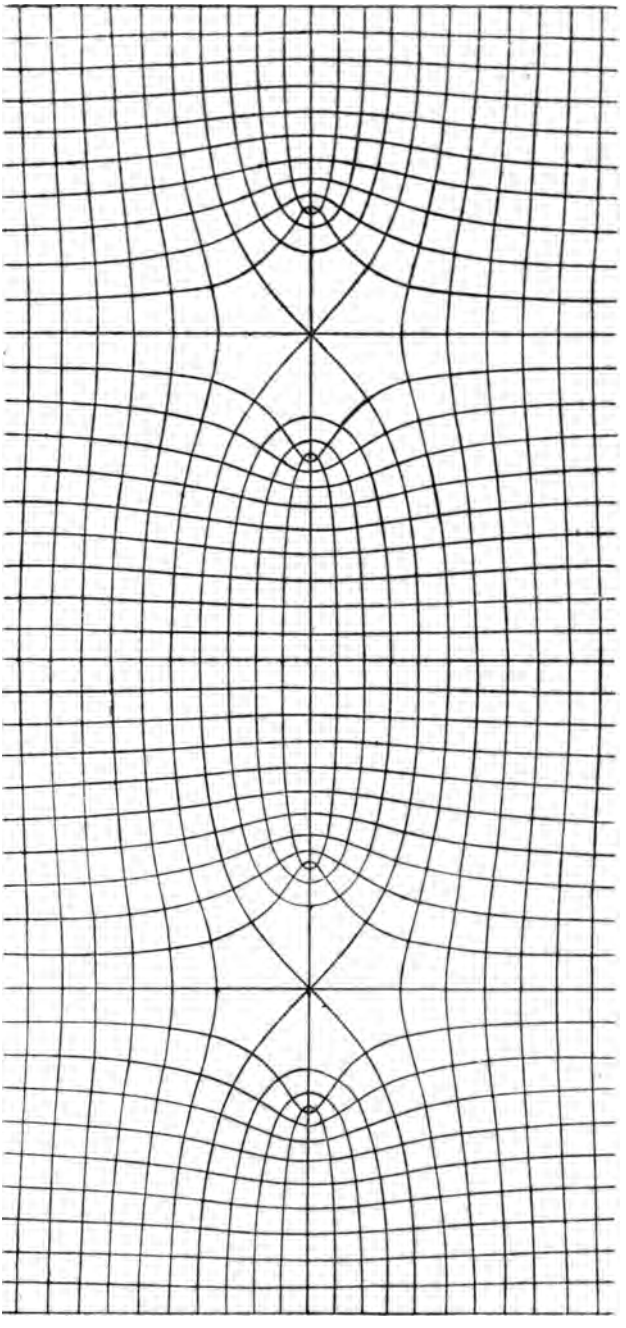

FIG. 1.

The formulæ (3), and the diagram, admit of the usual variety of interpretations in Electrostatics, Electric Conduction, Hydrodynamics, &c. If we introduce a more general linear unit, and denote the breadth of each aperture by a, that of each strip by b, we may write

$$\left.\begin{aligned} \cosh\phi\cos\psi &= \mu\cosh\frac{\pi x}{a+b}\cos\frac{\pi y}{a+b} \\ \sinh\phi\sin\psi &= \mu\sinh\frac{\pi x}{a+b}\sin\frac{\pi y}{a+b} \end{aligned}\right\}, \tag{6}$$

where

$$\mu = \sec\frac{\pi a}{2(a+b)} = \operatorname{cosec}\frac{\pi b}{2(a+b)}. \tag{7}$$

A first application of (6) is to find the capacity of a condenser formed by two metal plates at the same potential with an insulated grating half way between them, the distance between the two external plates being large compared with the interval $a+b$. For large positive values of x, the formulæ (6) give

$$\phi = \frac{\pi x}{a+b} + \log\mu, \quad \psi = \frac{\pi y}{a+b}. \tag{8}$$

The charge per unit area of one of the plates is $1/4(a+b)$ in the present units, and the capacity of the system per unit area is, therefore

$$\frac{1}{2\pi\left\{x + \dfrac{a+b}{\pi}\log\sec\dfrac{\pi a}{2(a+b)}\right\}}, \tag{9}$$

where x is the half-distance between the external plates. If the central plate had been entire, the capacity would have been $1/2\pi x$; the effect of the apertures is therefore equivalent to increasing the distance between the central plate and each external plate by an amount c, given by

$$c = \frac{a+b}{\pi}\log\sec\frac{\pi a}{2(a+b)}. \tag{10}$$

The numerical values subjoined shew what a very large proportion of the central plate can be cut away without sensibly altering the capacity, so long as the period $(a+b)$ of the residual structure is large compared with the distance between the plates:

$$\frac{a}{a+b} = 0,\ \cdot1,\ \cdot2,\ \cdot3,\ \cdot4,\ \cdot5,\ \cdot6,\ \cdot7,\ \cdot8,\ \cdot9,\ 1{\cdot}0$$

$$\frac{c}{a+b} = 0,\ \cdot004,\ \cdot016,\ \cdot037,\ \cdot067,\ \cdot110,\ \cdot169,\ \cdot251,\ \cdot374,\ \cdot590,\ \infty.$$

For another interpretation of (6) we may imagine that a current is flowing along a metal strip, of breadth $a+b$, which is interrupted by a central transverse non-conducting partition (*e.g.*, a narrow cut in the metal), of breadth a, or by two lateral transverse cuts, opposite to one another, each of breadth $\frac{1}{2}a$. It appears that the additional resistance thus introduced is equal to that of a length $2c$ of the strip.

2. The function ϕ in (3) is a periodic even function of y, the period being π. It can therefore be expanded by Fourier's Theorem in a series of cosines of multiples of $2y$, the coefficients being functions of x whose general form is to be determined by substitution in the equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0. \tag{11}$$

Thus, for x positive we find, having regard to the conditions to be satisfied for large values of x,

$$\phi=\log\mu+x+\Sigma_1^\infty C_m e^{-2mx}\cos 2my. \tag{12}$$

The precise values of the coefficients C_m hardly concern us, but it may be worth while to record them. The expansion of the function

$$w=\cosh^{-1}(\mu\cosh z) \tag{13}$$

can be effected directly, with the assistance of Lagrange's Theorem, and in this way we obtain the form (12), with

$$C_m=(-)^{m-1}\frac{1}{m}\left\{1-\frac{m^2}{1^2.\mu^2}+\frac{m^2(m^2-1^2)}{1^2.2^2.\mu^4}-\frac{m^2(m^2-1^2)(m^2-2^2)}{1^2.2^2.3^2.\mu^6}+\dots\right\}. \tag{14}$$

It may be noted that this contains $1-1/\mu^2$ as a factor.*

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* [*July*, 1898.—We have

$$C_m=(-)^{m-1}\frac{1}{m}F\left(m,-m,1,\frac{1}{\mu^2}\right)$$

$$=(-)^{m-1}\frac{1}{m}\left(1-\frac{1}{\mu^2}\right)F\left(1+m,1-m,1,\frac{1}{\mu^2}\right),$$

in the hypergeometric notation.

At the suggestion of one of the referees the proof of (14) is indicated. If we put $y=0$ in (13), we have

$$\phi=\cosh^{-1}(\mu\cosh x)=\log\{\mu\cosh x+\sqrt{(\mu^2\cosh^2 x-1)}\}$$

$$=\log\mu+x+\log\left[\frac{1+t}{2}+\sqrt{\left\{\left(\frac{1+t}{2}\right)^2-\frac{1}{\mu^2}\right\}}\right],$$

where $t=e^{-2x}$. If we write

$$\zeta=\frac{1+t}{2}+\sqrt{\left\{\left(\frac{1+t}{2}\right)^2-\frac{1}{\mu^2}\right\}},$$

The comparison of (12) with (3) leads to a number of results in definite integrals. For example, using square brackets to denote that x has been put $= 0$ in the values of the enclosed functions, we have

$$\int_0^{\frac{1}{2}\pi} [\phi]\, dy = \tfrac{1}{2}\pi \log \mu, \tag{15}$$

$$\int_0^{\frac{1}{2}\pi} \left[\frac{d\phi}{dx}\right] dy = \tfrac{1}{2}\pi. \tag{16}$$

Now it appears from (3) that from $y = 0$ to $y = \cos^{-1}(1/\mu)$ we have

$$[\psi] = 0, \quad [\phi] = \cosh^{-1}(\mu \cos y),$$

whilst, from $y = \cos^{-1}(1/\mu)$ to $y = \frac{1}{2}\pi$,

$$[\phi] = 0, \quad [\psi] = \cos^{-1}(\mu \cos y).$$

Hence (15) gives the theorem

$$\int_0^{\beta} \cosh^{-1}\left(\frac{\cos y}{\cos \beta}\right) dy = \tfrac{1}{2}\pi \log \sec \beta. \tag{17}$$

This is easily verified. If we denote the value of the definite integral on the left hand by $f(\beta)$, we find

$$f'(\beta) = \tan \beta \int_0^{\beta} \frac{\cos y\, dy}{\sqrt{(\cos^2 y - \cos^2 \beta)}} = \tfrac{1}{2}\pi \tan \beta, \tag{18}$$

whence (17) follows by integration, since $f(0) = 0$.* The verification of (16) is still simpler; we have only to put $d\phi/dx = d\psi/dy$.

---

we find $$\zeta = 1 + t\left(1 - \frac{1}{\mu^2 \zeta}\right) = 1 + t f(\zeta),$$

say. If x be positive, t will be less than unity, and $\log \zeta$ can be expanded in powers of t by Lagrange's Theorem, thus

$$\log \zeta = \log u + \Sigma_1^{\infty} \frac{t^m}{m!} \left(\frac{d}{du}\right)^{m-1} \frac{\{f(u)\}^m}{u},$$

where u is to be put $= 1$, finally. Working this out, we find

$$\log \zeta = \Sigma_1^{\infty} C_m t^m,$$

where C_m has the value given above. Hence the value of ϕ for $y = 0$ and $x > 0$ is

$$\phi = \log \mu + x + \Sigma_1^{\infty} C_m e^{-2mx}.$$

It easily follows that (12) gives the general value of ϕ for $x > 0$.]

* The theorem (17) is, substantially, one of a number given by Lobatschewsky, *Kasan Memoirs*, 1836 (Part I., p. 149), but the proof is quite different. The formula

$$\int_0^{\beta} \frac{y \sin y\, dy}{\sqrt{(\sin^2 \beta - \sin^2 y)}} = \tfrac{1}{2}\pi \log \sec \beta,$$

which is immediately deduced from (17) by a partial integration, was obtained by Legendre, *Exercices de Calcul Intégral*, t. I., Supplément, Case ix. (1811), as a particular case of a general theorem involving elliptic integrals.

3. When the field of force is not symmetrical with respect to the plane of the grating, the requisite formulæ are obtained by the addition of a term Az to w in (1), so that

$$\cosh(w+Az) = \mu\cosh z. \tag{19}$$

The most interesting special case is where the constant A is so chosen as to annul the force at a distance on one side, say that for which x is negative. We then have $A=-1$, and the equation (19) reduces to the form

$$\tanh z = \frac{\cosh w-\mu}{\sinh w}, \tag{20}$$

whence

$$z = \tfrac{1}{2}\log\frac{e^w-\mu}{\mu-e^{-w}}. \tag{21}$$

If we write $\mu = e^\alpha$, this can be put into the shape

$$z = \tfrac{1}{2}w+\tfrac{1}{2}\log\frac{\sinh\frac{1}{2}(w-\alpha)}{\sinh\frac{1}{2}(w+\alpha)}, \tag{22}$$*

or

$$x+iy = \tfrac{1}{2}(\phi+i\psi)+\tfrac{1}{2}\log\frac{\sinh\frac{1}{2}(\phi-\alpha)\cos\frac{1}{2}\psi+i\cosh\frac{1}{2}(\phi-\alpha)\sin\frac{1}{2}\psi}{\sinh\frac{1}{2}(\phi+\alpha)\cos\frac{1}{2}\psi+i\cosh\frac{1}{2}(\phi+\alpha)\sin\frac{1}{2}\psi}. \tag{23}$$

This gives, after a little manipulation,

$$\left.\begin{aligned} x &= \tfrac{1}{2}\phi+\tfrac{1}{4}\log\frac{\cosh(\phi-\alpha)-\cos\psi}{\cosh(\phi+\alpha)-\cos\psi}, \\ y &= \tfrac{1}{2}\psi+\tfrac{1}{2}\tan^{-1}\frac{\sinh\alpha\sin\psi}{\cosh\phi-\cosh\alpha\cos\psi} \end{aligned}\right\}. \tag{24}$$

By means of these equations the curves

$$\phi = \text{const.}, \quad \psi = \text{const.}$$

can be traced without much difficulty. Their forms are shown in Fig. 2 (p. 530) for the case of $\alpha = \frac{1}{5}\pi$. The diagram affords the best verification of the formulæ, but the following points may be noticed.

In the first place, the line $\psi = 0$ consists of distinct portions, according as $\phi \gtrless \alpha$. For $\phi > \alpha$ we may take

$$y = 0, \quad x = \tfrac{1}{2}\phi+\tfrac{1}{2}\log\frac{\sinh\frac{1}{2}(\phi-\alpha)}{\sinh\frac{1}{2}(\phi+\alpha)}, \tag{25}$$

the value of x ranging from $-\infty$ to $+\infty$ as ϕ increases from α to ∞. This gives the line of force through the origin, which is the centre of

* The solution for the present case was originally obtained in this form by Schwarz's method.

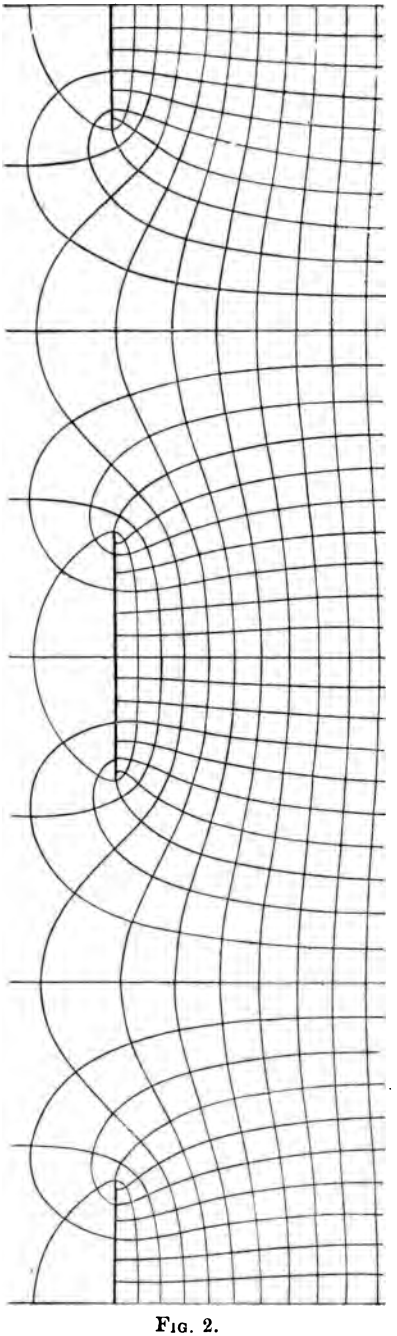

FIG. 2.

an aperture, and the lines of force through the centres of consecutive apertures are given in like manner by

$$y = \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \ldots,$$

corresponding to

$$\psi = \pm 2\pi, \quad \pm 4\pi, \quad \pm 6\pi, \quad \ldots.$$

If, on the other hand, $\psi = 0$ and $\phi < \alpha$, we have

$$y = \tfrac{1}{2}\tan^{-1} 0, \quad x = \tfrac{1}{2}\phi + \tfrac{1}{2}\log\frac{\sinh\tfrac{1}{2}(\alpha-\phi)}{\sinh\tfrac{1}{2}(\alpha+\phi)}. \tag{26}$$

The value of x ranges from 0 to $-\infty$ as ϕ increases from 0 to α. The value to be attributed to $\tan^{-1} 0$ is determined by considerations of continuity. Thus, if we assign a very small fixed positive value to ψ, and then imagine ϕ to decrease from ∞ through α to 0, the fraction

$$\frac{\sinh\alpha\sin\psi}{\cosh\phi - \cosh\alpha\cos\psi}$$

will pass from a small positive value through infinity to a small negative value. Hence, if ψ approach zero from the positive side, the value of y in (26) will be $\frac{1}{2}\pi$. If ψ were to approach zero from the negative side, we should have $y = -\frac{1}{2}\pi$. The portions of the lines

$$y = \pm\tfrac{1}{2}\pi, \quad \pm\tfrac{3}{2}\pi, \quad \pm\tfrac{5}{2}\pi, \quad \ldots$$

for which x is negative are lines of force abutting on the backs of the metal strips.

Again, for $\psi = \pi$, we have

$$y = \tfrac{1}{2}\pi, \quad x = \tfrac{1}{2}\phi + \tfrac{1}{2}\log\frac{\cosh\tfrac{1}{2}(\phi-\alpha)}{\cosh\tfrac{1}{2}(\phi+\alpha)}; \tag{27}$$

this value of x ranges from 0 to ∞ as ϕ increases from 0 to ∞. And, generally, the portions of the lines

$$y = \pm\tfrac{1}{2}\pi, \quad \pm\tfrac{3}{2}\pi, \quad \pm\tfrac{5}{2}\pi, \quad \ldots$$

for which x is positive are lines of force abutting on the fronts of the metal strips.

If in (24) we make $\phi = 0$, we get

$$x = 0, \quad y = \tfrac{1}{2}\psi + \tfrac{1}{2}\tan^{-1}\frac{\sinh\alpha\sin\psi}{1-\cosh\alpha\cos\psi}. \tag{28}$$

This gives a series of isolated portions of the axis of y, representing the breadths of the metal strips. These are bounded by the points for which

$$\cos\psi = e^{-\alpha} = 1/\mu,$$

and therefore

$$\frac{\sinh\alpha\sin\psi}{1-\cosh\alpha\cos\psi} = \tan\psi. \tag{29}$$

For example, if we make ψ increase from 0 to π, y will diminish from $\frac{1}{2}\pi$ to the minimum value $\cos^{-1}e^{-a}$, and then increase again up to the value $\frac{1}{2}\pi$.

It is further to be noticed that the lines $\phi = \text{const.}$ fall into two systems separated by the curves $\phi = a$, which have the lines

$$y = 0, \ \pm\pi, \ \pm 2\pi, \ \ldots$$

as asymptotes. For $\phi < a$, the equipotential lines are closed. This is indicated clearly in the figure, which may be compared with the corresponding diagram for a grating of cylindrical wires given by Maxwell.*

The formulæ (24), when generalized as to the linear unit involved, lead to a determination of the capacity of a condenser formed of two parallel plates, one of which is entire, whilst the other is a grating. If the distance x between these plates be large compared with the interval $a+b$, where a and b have the same meanings as in § 1, we find for the capacity in question

$$\frac{1}{4\pi\left\{x + \frac{a+b}{2\pi}\log\sec\frac{\pi a}{2(a+b)}\right\}}. \qquad (30)$$

It is also easy to shew that the charge on the grating is divided between its front and back surfaces in the ratio $a+2b : a$.

4. Although the matter is not very closely connected with the main object of this paper, it may be worth while to notice the interpretation of the formulæ (3) when the constant μ is *less* than unity. They give the distribution of electric force in a condenser one of whose plates has a series of parallel equidistant ribs projecting at right angles to its surface. Thus, if, altering the linear unit, we write

$$\cosh w = \mu \cosh\frac{\pi z}{a}, \qquad (31)$$

or

$$\left.\begin{aligned} \cosh\phi\cos\psi &= \mu\cosh\frac{\pi x}{a}\cos\frac{\pi y}{a} \\ \sinh\phi\sin\psi &= \mu\sinh\frac{\pi x}{a}\sin\frac{\pi y}{a} \end{aligned}\right\}, \qquad (32)$$

we find that the line $\phi = 0$ now consists of the whole of the axis of y, together with those portions of the lines

$$y = 0, \ \pm a, \ \pm 2a, \ \pm 3a, \ \ldots$$

which extend from the axis of y to the points determined by

$$\cosh\frac{\pi x}{a} = \frac{1}{\mu}. \qquad (33)$$

* *Electricity and Magnetism*, Vol. I., Fig. xiii.

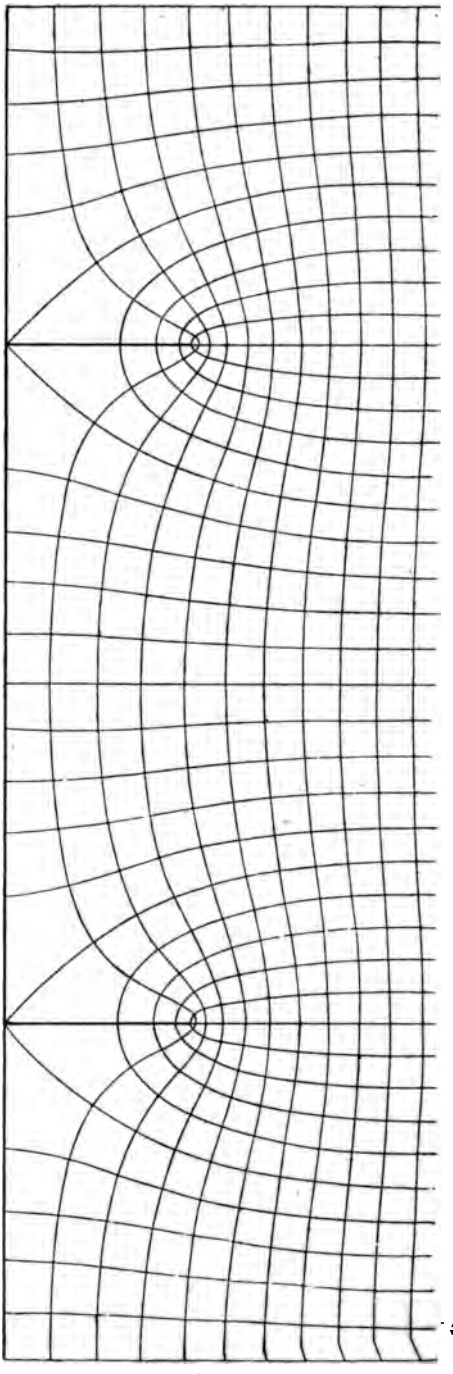

FIG. 3.

Hence, if b be the depth of a rib, we have

$$\mu = \operatorname{sech} \frac{\pi b}{a}. \tag{34}$$

The curves $\phi = \text{const.}$, $\psi = \text{const.}$ are shown in Fig. 3, for the case of $\mu = 1/\sqrt{2}$.

Proceeding as in § 1, we find that the capacity (per unit area) of a condenser of the above type is

$$\frac{1}{4\pi\left(x - \frac{a}{\pi} \log \cosh \frac{\pi b}{a}\right)}, \tag{35}$$

x denoting the distance between the plates, which is assumed to be large compared with b. The effect of the ribs in increasing the capacity is therefore equivalent to diminishing the distance between the plates by the amount

$$\frac{a}{\pi} \log \cosh \frac{\pi b}{a}. \tag{36}$$

Again, if we take the curves $\phi = \text{const.}$ to be lines of electric flow, we get the case of a current flowing along a relatively broad strip which is interrupted along one edge, or along the medial line, by a series of parallel, equal, and equidistant transverse cuts, one behind the other. If b be the length of the cuts, their effect (when at the edge) is equivalent to diminishing the breadth of the strips by the amount (36). As numerical examples, we find, for

$$b/a = \cdot1, \quad \cdot2, \quad \cdot3, \quad \cdot4, \quad \cdot5, \quad \cdot6, \quad \cdot7, \quad \cdot8, \quad \cdot9, \quad 1{\cdot}0,$$

$$\frac{\log \cosh \pi b/a}{\pi b/a} = \cdot156, \ \cdot296, \ \cdot415, \ \cdot511, \ \cdot586, \ \cdot645, \ \cdot691, \ \cdot729, \ \cdot756, \ \cdot780.$$

The latter numbers give the ratio which the aforesaid diminution bears to the breadth b of the cuts. For increasing values of b/a this ratio tends obviously to the value unity.

In the case of cuts across the medial line the quantity (36) must be multiplied by 2, b now denoting the half-length of a cut.

5. Proceeding to the problem of electric waves, and assuming that all our functions involve the two dimensions x and y only, we have to satisfy an equation of the form

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + k^2u = 0, \tag{37}$$

subject to the proper condition at the surfaces of the metal. If the

electrical vibrations be sufficiently rapid, the condition is that the component of electric force tangential to the surface, just outside the metal, must be zero.

We take separately the cases where the direction of the electric force is perpendicular and parallel, respectively, to the plane xy. In the former case, the equation (37) is satisfied by the electric force (R), and the condition which is to hold at the metal surfaces is $R = 0$, simply.

Let us now suppose that a train of waves of length $2\pi/k$ and amplitude unity is incident directly on the grating from the side on which x is positive. Omitting the time-factor e^{ikVt}, where V is the wave-velocity, this train may be represented symbolically by

$$R = e^{ikx}. \tag{38}$$

If in place of the grating we had an uninterrupted sheet of metal, the reflected wave would be represented by $-e^{-ikx}$, and the total disturbance on the side of x positive would be given by

$$R = e^{ikx} - e^{-ikx}, \tag{39}$$

for this satisfies (37) and makes $R = 0$ for $x = 0$.

In the actual circumstances we assume that, on the positive side of the grating,

$$R = e^{ikx} - e^{-ikx} + \chi, \tag{40}$$

and, on the negative side,

$$R = \chi', \tag{41}$$

where χ and χ' must, of course, satisfy (37). Over the portions of the plane $x = 0$ which are occupied by the metal strips, we must have

$$\chi = 0, \quad \chi' = 0; \tag{42}$$

whilst the continuity of R and dR/dx demands that

$$\chi = \chi', \quad \frac{d\chi}{dx} + 2ik = \frac{d\chi'}{dx} \tag{43}$$

over the apertures. The latter condition will be satisfied provided we make

$$\frac{d\chi}{dx} = -ik, \quad \frac{d\chi'}{dx} = ik. \tag{44}$$

If the differential equation to be satisfied by χ had been

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \tag{45}$$

in place of (37), the conditions relating to this function would (save as to the factor ik) have been the same as in the hydrodynamical problem where a system of equal, parallel, and equidistant

blades, all in the same plane, is moved perpendicular to itself in a frictionless liquid, which may have an independent general motion in the same direction. The function χ/ik would, in fact, give the value of the velocity-potential on the positive side of the plane $x = 0$. It must be borne in mind in this analogy that the blades correspond to the *apertures* of our grating, and the intervals between them to the metallic strips.* Now the *relative* motion of the fluid in such a case is shown in Fig. 1, to which the formulæ (6) apply; and a value of χ satisfying the conditions (42) and (44) would therefore be

$$\chi = -ikx + C\phi, \tag{46}$$

where ϕ is the positive function determined by (6).

This value of χ is necessarily periodic with respect to y, the period being $a+b$; and the line $y = 0$ is a line of symmetry. Hence, expanding by Fourier's Theorem in a series of cosines of multiples of $2\pi y/(a+b)$, we find

$$\chi = C \log \mu + \left(\frac{\pi C}{a+b} - ik\right) x + C \Sigma_1^\infty C_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}, \tag{47}$$

where

$$\kappa_m = \frac{2m\pi}{a+b}, \tag{48}$$

and the first two terms have been determined from a comparison of (46) and (8). The terms involving $e^{\kappa_m x}$ are excluded by the condition of finiteness for $x = \infty$.

The true value of χ can be similarly expanded in a Fourier's series, the forms of the coefficients as functions of x being determined by means of the differential equation (37). The resulting form is

$$\chi = B_0 e^{-ikx} + \Sigma_1^\infty B_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a+b}, \tag{49}$$

provided

$$\lambda_m^2 = \frac{4m^2\pi^2}{(a+b)^2} - k^2. \tag{50}$$

Since, by hypothesis, $a+b$ is small compared with the wave-length $2\pi/k$, the right-hand side of (50) is positive. Hence the quantities λ_m are real, and, moreover, differ respectively very little from κ_m. Terms involving $e^{\lambda_m x}$ are excluded as before by the condition of finiteness for $x = \infty$, so that the waves represented by χ are ultimately plane. The fact that they must travel outwards from the grating justifies the omission of the term in e^{ikx}.

* The foregoing method of analysing the conditions to be satisfied is taken from Lord Rayleigh's paper, *Phil. Mag.*, April, 1897.

For small values of x, the expansion (49) reduces to the form

$$\chi = B_0(1-kx) + \Sigma_1^\infty B_m e^{-\kappa_m x}\cos\frac{2m\pi y}{a+b}, \qquad (51)^*$$

approximately; and it appears, on comparison with (47), that the conditions incumbent on χ will all be satisfied, subject to an error of the order $k^2(a+b)^2/4\pi^2$, if

$$B_m = CC_m, \quad B_0 = C\log\mu = 1+\frac{i\pi C}{k(a+b)}. \qquad (52)$$

Substituting the value of μ from (7), we find

$$B_0 = \frac{ikc}{1+ikc}, \qquad (53)$$

where c is the linear magnitude defined by (10).

The conditions for χ' are all fulfilled if we suppose that its value at any point P' on the negative side of the grating is equal, both in magnitude and in sign, to the value of χ at the *image* P of P' on the positive side. Hence the Fourier expansion of χ' is

$$\chi' = B_0 e^{ikx} + \Sigma_1 B_m e^{\lambda_m x}\cos\frac{2m\pi y}{a+b}. \qquad (54)$$

In the above investigation, the coefficient of the primary wave has been taken to be unity. On the same scale, the coefficients of the reflected and transmitted waves are $-1+B_0$ and B_0, or

$$-\frac{1}{1+ikc} \quad \text{and} \quad \frac{ikc}{1+ikc},$$

respectively. Hence the *intensities* I, I' (say) of these waves, in terms of that of the primary wave as unit, are given by

$$I = \frac{1}{1+k^2c^2}, \quad I' = \frac{k^2c^2}{1+k^2c^2}. \qquad (55)$$

It appears from the numerical values given near the end of § 1 that the value of $c/(a+b)$ may be very moderate even when the breadth (b) of the metal strips is relatively small. Hence even a very open grating (*i.e.*, one in which a very small proportion of the area is occupied by metal) will produce almost total reflection of waves in which the electric displacement is parallel to the bars, provided the wave-length ($2\pi/k$) be moderately large compared with the period $a+b$ of the structure.

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* *Cf.* J. J. Thomson, *loc. cit.*

The circumstances when the reflection is practically total are clearly illustrated in Fig. 2, where the lines $\phi = \text{const.}$ represent the lines of (alternating) magnetic force in the immediate neighbourhood of the grating. A similar interpretation applies also to Maxwell's diagram already referred to.

6. When the electric force is everywhere parallel to the plane xy, the magnetic force (γ) is everywhere perpendicular to this plane, and satisfies an equation of the form (37). The components of displacement-current in the dielectric are then

$$\frac{1}{4\pi}\frac{d\nu}{dy}, \quad -\frac{1}{4\pi}\frac{d\gamma}{dx}, \quad 0,$$

and the condition to be satisfied at the surfaces of the metal strips is $d\gamma/dn = 0$, where dn denotes an element of the normal.

Considering a train of waves represented by

$$\gamma = e^{ikx}, \tag{56}$$

and therefore incident directly on the grating from the side of x positive, we should have, if the apertures were filled up,

$$\gamma = e^{ikx} + e^{-ikx} \tag{57}$$

on the positive side, and $\gamma = 0$ on the negative side. In the actual case we assume, for the positive side,

$$\gamma = e^{ikx} + e^{-ikx} + \chi, \tag{58}$$

and, for the negative side, $\qquad \gamma = \chi', \tag{59}$

where χ and χ' must satisfy (37); and the conditions to be fulfilled over those portions of the line $x = 0$ which correspond to the metal strips are

$$\frac{d\chi}{dx} = 0, \quad \frac{d\chi'}{dx} = 0, \tag{60}$$

whilst over the apertures we must have

$$2 + \chi = \chi', \quad \frac{d\chi}{dx} = \frac{d\chi'}{dx}, \tag{61}$$

on account of the continuity of γ and $d\gamma/dx$. The former of the conditions (61) will be satisfied provided

$$\chi = -1, \quad \chi' = 1. \tag{62}$$

If the differential equation (37) were replaced by (45), the conditions applicable to χ would be similar to those which govern the

electric potential in the case of a grating placed transverse to the lines of force in a uniform field; but it must be remembered in this analogy that the bars in one problem correspond to the apertures in the other, and *vice versâ*.* We should therefore have

$$\chi = -1 + C\phi, \tag{63}$$

where ϕ is determined by (6); but, if we still continue to denote by a and b the breadths of the apertures and of the strips, respectively, in our grating, the formula (7), giving the value of μ, must be replaced by

$$\mu = \sec\frac{\pi b}{2(a+b)} = \operatorname{cosec}\frac{\pi a}{2(a+b)}. \tag{64}$$

The Fourier expansion of (63) is

$$\chi = -1 + C\log\mu + \frac{\pi C}{a+b}x + C\Sigma_1^\infty C_m e^{-\kappa_m x}\cos\frac{2m\pi y}{a+b}, \tag{65}$$

whilst that of the true function χ has the form

$$\chi = C_0 e^{-ikx} + \Sigma_1^\infty B_m e^{-\lambda_m x}\cos\frac{2m\pi y}{a+b}, \tag{66}$$

where κ_m and λ_m are defined by (48) and (50). For small values of x the latter expansion takes the form

$$\chi = C_0(1-ikx) + \Sigma_1^\infty B_m e^{-\kappa_m x}\cos\frac{2m\pi y}{a+b}. \tag{67}$$

By comparison we find that (66) will satisfy approximately all the conditions of the problem, provided

$$B_m = CC_m, \quad C_0 = -1 + C\log\mu = \frac{i\pi C}{k(a+b)}. \tag{68}$$

Hence
$$C_0 = -\frac{1}{1+ikc_1}, \tag{69}$$

provided
$$c_1 = \frac{a+b}{\pi}\log\sec\frac{\pi b}{2(a+b)}. \tag{70}$$

As regards χ', all the conditions are satisfied if we suppose that its value at any point P' on the negative side of the grating is equal in absolute magnitude, but opposite in sign, to that of χ at the image P of P' on the positive side. Hence the Fourier expansion of χ' is

$$\chi' = -C_0 e^{ikx} - \Sigma_1^\infty B_m e^{\lambda_m x}\cos\frac{2m\pi y}{a+b}. \tag{71}$$

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* See the footnote on p. 536.

The coefficients of the reflected and transmitted waves are $1+C_0$ and $-C_0$, or

$$\frac{ikc_1}{1+ikc_1} \quad \text{and} \quad \frac{1}{1+ikc_1},$$

respectively, that of the primary being taken as unit. Hence the intensities J, J' of these waves are given by

$$J = \frac{k^2c_1^2}{1+k^2c_1^2}, \qquad J' = \frac{1}{1+k^2c_1^2}. \tag{72}$$

For sufficiently great wave-lengths there is very little reflection, even when the intervals a between the strips are very narrow compared with the breadths of the strips themselves.

* [These results, combined with those of § 5, serve to explain the polarizing efficacy of a grating, as employed originally by Hertz. As a numerical example, let us suppose that the wave-length is ten times the interval $a+b$. Then, even if one-tenth only of the area of the grating be occupied by the metal strips, the coefficients of reflection and transmission are, for electrical vibrations parallel to the lengths,

$$I = \cdot 879, \quad I' = \cdot 121.$$

The same numbers in reverse order give the coefficients of reflection and transmission for electrical vibrations transverse to the length, when the *apertures* occupy only one-tenth of the area; viz.,

$$J = \cdot 121, \quad J' = \cdot 879.$$

The coefficient of reflection in the one case, and of transmission in the other, increases rapidly with the wave-length.]

In the immediate neighbourhood of the grating the lines $\gamma = \text{const.}$ coincide with the curves $\phi = \text{const.}$ of Fig. 1. These give the directions of the electric oscillation; but it must be remembered that the portions of the vertical line of symmetry which represented the metal strips in the electrostatic problem of § 1 now correspond to apertures, and *vice versâ*.

The mathematical conditions to which the magnetic force γ is subject in the problem of this section are exactly those which apply to the velocity-potential in the case of sound-waves incident directly on a grating of the kind considered.† It appears that a series of narrow slits, with relatively broad opaque intervals, may transmit a *very* large proportion of the original sound, provided the wave-length be sufficiently great.

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* Paragraph inserted July, 1898.

† *Cf.* Rayleigh, *Phil. Mag.*, April, 1897, p. 272.

7. As mentioned in the introduction, the problem of reflection by a grating composed of parallel cylindrical bars has been discussed by Prof. J. J. Thomson in the case where the electric displacement is everywhere in the direction of the length. The result arrived at is that the reflection is total, without any limitation as to the thinness of the bars. It may be worth while to examine this point, and to complete the investigation by considering also the case where the electric displacement is transverse.

If, as in § 1, we write

$$w = \phi + i\psi, \quad z = x + iy, \tag{73}$$

the potential- and stream-functions corresponding to a system of equal and equidistant line-sources cutting the plane xy perpendicularly at the points $(0, 0)$, $(0, \pm a)$, $(0, \pm 2a)$, ..., are given by the formula

$$w \propto \log z + \log(z - ia) + \log(z + ia) + \log(z - 2ia) + \log(z + 2ia) + \ldots, \tag{74}$$

or, say,

$$w = \log \sinh \frac{\pi z}{a}. \tag{75}$$

This makes

$$\left.\begin{aligned} \phi &= \frac{1}{2}\log\frac{1}{2}\left(\cosh\frac{2\pi x}{c} - \cos\frac{2\pi y}{c}\right) \\ \psi &= \tan^{-1}\left\{\frac{\tan\frac{\pi y}{c}}{\tanh\frac{\pi x}{a}}\right\} \end{aligned}\right\}, \tag{76}$$

in agreement with Maxwell, *Electricity and Magnetism*, § 203.

The case of a row of *double* sources having their axes parallel to x is obtained by differentiating (75) with respect to z; thus we find

$$w = \frac{\pi}{a}\log\coth\frac{\pi z}{a}, \tag{77}$$

$$\left.\begin{aligned} \phi &= \frac{\pi}{a}\,\frac{\sinh\frac{2\pi x}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}} \\ \psi &= -\frac{\pi}{a}\,\frac{\sin\frac{2\pi y}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}} \end{aligned}\right\}. \tag{78}$$

If the double sources have their axes parallel to y, we have merely to interchange the values of ϕ and ψ in these formulæ.

These results enable us to solve very simply a number of problems in Hydrodynamics, Magnetic Induction, and so on. For example, the stream-function for a liquid flowing through a grating of parallel cylindrical bars is

$$\psi = V\left\{y - \frac{\pi b^2}{a}\frac{\sin\frac{2\pi y}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}}\right\}, \qquad (79)^*$$

where V is the velocity at infinity. This formula is easily verified, since, for small values of x and y, it reduces to the form

$$\psi = Vy\left(1 - \frac{b^2}{x^2+y^2}\right). \qquad (80)$$

It is here assumed that the radius b of a bar is small compared with the distance a between the axes of consecutive bars.

Again, in the case of a liquid flowing *parallel* to a row of cylindrical obstacles, we have

$$\psi = V\left\{x - \frac{\pi b^2}{a}\frac{\sinh\frac{2\pi x}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}}\right\}. \qquad (81)$$

Returning to the formula (75), we find, if the real part of z be positive,

$$w = \log\tfrac{1}{2}\left(e^{\pi z/a} - e^{-\pi z/a}\right)$$

$$= \frac{\pi z}{a} - \log 2 - \Sigma_1^\infty \frac{1}{m} e^{-2m\pi z/a}, \qquad (82)$$

whence

$$\phi = \frac{\pi x}{a} - \log 2 - \Sigma_1^\infty \frac{1}{m} e^{-2m\pi x/a}\cos\frac{2m\pi y}{a}. \qquad (83)$$

Since ϕ is an even function of x, we have, for x negative,

$$\phi = -\frac{\pi x}{a} - \log 2 - \Sigma_1^\infty \frac{1}{m} e^{2m\pi x/a}\cos\frac{2m\pi y}{a}. \qquad (84)$$

For small values of x, we have, from (76),

$$\phi = \log\frac{\pi r}{a}, \qquad (85)$$

where

$$r = \sqrt{(x^2+y^2)}.$$

When the electric force (R) is everywhere parallel to the bars of the grating, we assume, for x positive,

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* The curves ψ = const. have been traced from this formula by Prof. Hele Shaw, *Trans. Inst. Nav. Arch.*, 1898.

$$R = e^{ikx} + Ae^{-ikx} + \Sigma_1^\infty D_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{86}$$

and, for x negative,

$$R = Be^{ikx} + \Sigma_1^\infty D_m e^{\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{87}$$

where λ_m is determined by

$$\lambda_m^2 = \frac{4m^2\pi^2}{a^2} - k^2. \tag{88}$$

For small values of x, we may ignore the difference between λ_m and $2m\pi/a$; and, if we further put $D_m = -C/m$, the formulæ (86) and (87) may be put into the shapes

$$R = 1 + A + ik(1-A)x + C\left(\phi - \frac{\pi x}{a} + \log 2\right), \tag{89}$$

and

$$R = B + ikBx + C\left(\phi + \frac{\pi x}{a} + \log 2\right), \tag{90}$$

respectively, where ϕ is defined by (83) and (84). The continuity of R and dR/dx requires that

$$1 + A = B, \quad ik(1-A) - \frac{\pi}{a}C = ikB + \frac{\pi}{a}C. \tag{91}$$

At the surface of the metal wires, where $r = b$ (say), we must have $R = 0$. This cannot be satisfied exactly; but, if the ratio b/a be sufficiently small, it is satisfied approximately if we make

$$B + C \log \frac{2\pi b}{a} = 0. \tag{92}*$$

If we now write

$$c = \frac{a}{\pi} \log \frac{a}{2\pi b}, \tag{93}$$

we find, from (91) and (92), that

$$A = -\frac{1}{1+ikc}, \quad B = \frac{ikc}{1+ikc}. \tag{94}$$

The intensities of the reflected and transmitted waves, in terms of that of the primary wave, are therefore

$$= \frac{1}{1+k^2c^2}, \quad I' = \frac{k^2c^2}{1+k^2c^2}. \tag{95}$$

If the wave-length is at all large compared with c, kc is small, and the reflection is almost total. But, for any given wave-length (large compared with a), kc may be made as great as we please by sufficiently diminishing the radius b of the wires. In this way we can pass to the case of free transmission.

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* [*July*, 1898.—Viz., this makes $R = 0$ for $x = 0$, $r = b$.]

When the electric force is everywhere parallel to the plane xy, it is convenient (as in § 6) to adopt as dependent variable the magnetic force (γ), which is everywhere perpendicular to this plane. We assume, for x positive,

$$\gamma = e^{ikx} + Ae^{-ikx} + \Sigma E_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{96}$$

and, for x negative,

$$\gamma = Be^{ikx} - \Sigma E_m e^{\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{97}$$

where λ_m is defined by (88). If we put $E_m = 2\pi C/a$, and ignore the difference between $\lambda_m x$ and $2m\pi x/a$, this gives, for small values of x,

$$\gamma = 1 + A + ik(1-A)x + C\left(\frac{d\phi}{dx} - \frac{\pi}{a}\right), \tag{98}$$

and

$$\gamma = \quad B + ikBx + C\left(\frac{d\phi}{dx} + \frac{\pi}{a}\right), \tag{99}$$

respectively. The continuity of γ and $d\gamma/dx$ on the two sides of the plane $x = 0$ requires that

$$1 + A - \frac{\pi}{a}C = B + \frac{\pi}{a}C, \quad 1 - A = B. \tag{100}$$

The surface-condition is that $d\gamma/dr = 0$ for $r = b$. If we put $x = r\cos\theta$, $d\phi/dx = \cos\theta/r$, this gives

$$ikB - \frac{C}{b^2} = 0. \tag{101}$$

Hence, if we write

$$c_1 = \frac{\pi b^2}{a}, \tag{102}$$

we get

$$A = \frac{ikc_1}{1+ikc_1}, \qquad B = \frac{1}{1+ikc_1}. \tag{103}$$

The intensities of the reflected and transmitted waves are now

$$J = \frac{k^2c_1^2}{1+k^2c_1^2}, \qquad J' = \frac{1}{1+k^2c_1^2}. \tag{104}$$

If the half wave-length be large compared with b^2/a, we have free transmission, with hardly any reflection.

As in the problem of § 6, this last calculation has also an acoustical interpretation, and serves to further illustrate the "extreme smallness of the obstruction offered by fine wires or fibres to the passage of sound." *

* Rayleigh, *Sound*, Vol. II., § 343.

Thursday, June 9th, 1898.

Prof. E. B. ELLIOTT, F.R.S., President, in the Chair.

Eleven members present and a visitor (Prof. F. Morley).

Prof. H. A. Lorentz, of Leyden, and M. Emile Picard, of Paris, were elected (Honorary) Foreign Members; and Mr. John Henry Michell, M.A., late Fellow of Trinity College, Cambridge, Assistant Professor of Mathematics in the University of Melbourne, was elected an ordinary member.

The President briefly announced the recent deaths of Mr. Henry Perigal (elected January 23rd, 1868), and of the Rev. Dr. Percival Frost, F.R.S. (elected December 9th, 1869).

The following papers were read by their authors:—

Point Groups in a Plane, and their effect in determining Algebraic Curves: Mr. F. S. Macaulay.

On a Regular Rectangular Configuration of Ten Lines: Prof. F. Morley.

On the Conformal Representation of a Pentagon on a Half Plane: Miss M. E. Barwell.

Messrs. Berry, Cunningham, Burnside, Love, the Chairman, and Miss Hardcastle took part in discussions upon one or more of the papers.

The following communications were made in abstract:—

On the General Theory of Anharmonics: Prof. E. O. Lovett.

On the Calculus of Equivalent Statements (Eighth Paper): Mr. H. MacColl.

On the Continuous Group that is defined by any given Group of Finite Order (Second Paper): Prof. W. Burnside.

On certain Regular Polygons in Modular Network: Prof. L. J. Rogers.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 4, 5.

"Proceedings of the Royal Society," Vol. LXIII., Nos. 393–95.

"Proceedings of the Royal Society of Edinburgh," Vol. XXI., 1895–6, 1896–7.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLII., Pt. 2; Manchester, 1897–98.

"Bulletin des Sciences Mathématiques," Tome XXI., "Tables des Matières": Tome XXII., Mai, 1898; Paris.

"Bulletin of the American Mathematical Society," Vol. IV., No. 8, May, 1898; New York.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 1–23, 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 9, 10, Sem. 1; Roma, 1898.

"Educational Times," June, 1898.

"Journal für die reine und angewandte Mathematik," Bd. CXIX., Heft 2; Berlin, 1898.

"Indian Engineering," Vol. XXIII., Nos. 17–20; April 23–May 14, 1898.

Whitehead, A. N.—"Treatise on Universal Algebra," Vol. I., 4to; Cambridge, 1898.

Frolov, M.—La Théorie des Parallèles," pamphlet, 8vo; Paris, 1898.

Rayet, G.—"Observations Pluviométriques et Thermométriques" de Juin 1896 à Mai 1897; Bordeaux, 1897.

"Procès-Verbaux des Séances de la Société des Sciences Physiques et Naturelles de Bordeaux," 1896-7; Paris.

"Journal de l'Ecole Polytechnique de Paris," Série II., Cahier 2; Paris, 1897.

On the Continuous Group that is defined by any given Group of Finite Order. (Second Paper.) By W. Burnside. Received June 8th, 1898. Read June 9th, 1898.

In the present paper the analysis of the continuous group G defined by the equations

$$x'_s = \Sigma y_{s-1i} x_i \quad (s,\ i = 1,\ 2,\ \ldots,\ n)$$

(where

$$y_{pq} = y_t,$$

if

$$S_p S_q = S_t;$$

$S_1 (= 1)$, S_2, ..., S_n being the operations of a discontinuous group g of finite order n) is brought to a definite conclusion. In a former paper with the same title,* it is shown that that group is the direct product of an Abelian group of order r and a semi-simple group of order $n-r$, r being the number of conjugate sets of operations in g.

It is here shown that r integers

$$\mu_1 (= 1),\ \mu_2,\ \ldots,\ \mu_r$$

exist such that

$$\mu_1^2 + \mu_2^2 + \ \ldots\ + \mu_r^2 = n,$$

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* *Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 207–224.

and that G is the direct product of r groups

$$G_1, G_2, \ldots, G_r,$$

which are simply isomorphic with the general linear homogeneous groups in

$$\mu_1, \mu_2, \ldots, \mu_r$$

variables respectively. Incidentally it is also shown that the determinant formed from the multiplication table of g, when in it the symbols of the operations are regarded as independent variables, has r distinct irreducible factors; and that the power to which any one of these factors enters into the determinant is equal to the degree of the factor. These are two of the main results proved by Herr Frobenius in his memoir "Über die Primfactoren der Gruppendeterminante" (*Berliner Sitzungsberichte*, 1896, pp. 1343–1382).

It is further shown that, if g can be represented as a group of linear substitutions performed on m symbols, so that to the operation S_k of g there corresponds the substitution

$$x_s' = \sum_{i=1}^{i=m} a_{sik} x_i \quad (s = 1, 2, \ldots, m),$$

then the equations

$$x_s' = \sum_{i,k} a_{sik} x_i y_k \quad (s, i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$$

define a finite continuous group which is the direct product of a number of groups each of which is simply isomorphic with a general linear homogeneous group.

The simultaneous consideration of the discontinuous group g, and of the continuous group G that is allied to it, appears to me likely to facilitate the discussion of certain of the properties of g. In particular, it obviously has a bearing on the question of the smallest number of variables in which g can be expressed as a group of linear substitutions, *i.e.*, on what Prof. Klein calls the degree of the normal-problem connected with g. Again, the properties of the group-determinant above mentioned, when investigated in this connexion, appear in a particularly clear and simple light.

In a first section are given some properties of the characteristic determinant of a simply transitive linear homogeneous group which are required for a discussion of G. These are not intended to be exhaustive; and some of them, including the contents of § 6, are equivalent to results obtained by Herr Molien in his memoir "Über Systeme höherer complexer Zahlen" (*Math. Ann.*, XLI., pp. 83–156, 1892). The form in which the properties are here presented and the methods of proof are quite distinct from those of Herr Molien.

[*Sept.* 19*th*, 1898.—Since communicating this paper to the Society I have seen a further memoir by Herr Frobenius, "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen," *Berliner Sitzungsberichte*, 1897, pp. 994–1015, which deals with the questions discussed here in §§ 11–13. In it reference is made to a memoir by Herr Molien, "Eine Bemerkung zur Theorie der homogenen Substitutionsgruppen," *Sitzungsberichte der Naturforscher-Gesellschaft zu Dorpat*, 1897, discussing the same question. This memoir I have not been able to gain access to.]

1.

1. The equations of a simply transitive linear homogeneous group G in n variables can always be written in the form

$$x'_s = \sum_{i,k} a_{sik} x_i y_k \quad (s,\ i,\ k = 1,\ 2,\ \ldots,\ n).$$

In my paper "On Linear Homogeneous Groups whose Operations are Permutable" I have shown that, by making, if necessary, a linear transformation of variables and parameters, the coefficients may be taken such that

$$a_{s1k} = 0,\ a_{sk1} = 0,\ s \neq k;\ a_{s1s} = 1;\ a_{ss1} = 1.$$

When these conditions are satisfied, G is said to be in "standard form"; and the equations of G', the reciprocal group, are

$$x'_s = \sum_{i,k} a_{ski} x_i y_k \quad (s,\ i = 1,\ 2,\ \ldots,\ n).$$

2. The characteristic determinant of the general finite operation of G is

$$\begin{vmatrix} \Sigma a_{11k} y_k + \lambda, & \Sigma a_{12k} y_k, & \ldots, & \Sigma a_{1nk} y_k \\ \Sigma a_{21k} y_k, & \Sigma a_{22k} y_k + \lambda, & \ldots, & \Sigma a_{2nk} y_k \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma a_{n1k} y_k, & \Sigma a_{n2k} y_k, & \ldots, & \Sigma a_{nnk} y_k + \lambda \end{vmatrix}.$$

If G is, as supposed, in standard form, the only elements in which y_1 occurs are those of the leading diagonal, and in each of these it occurs with the coefficient unity. The characteristic determinant of the general finite operation of the group is therefore the same as the determinant of the finite equations of the group with respect to the x's if in the latter y_1 be replaced by $y_1 + \lambda$. Now the determinant of the finite equations of the group

$$x'_s = \Sigma a_{sik} x_i y_k,$$

with respect to the x's, is the same as that of the finite equations

$$x'_s = \Sigma a_{ski} x_i y_k$$

of the reciprocal group, with respect to the y's. If, then, we consider simultaneously G and its reciprocal group G', the characteristic determinant of the general operation of either is obtained from the determinant of the finite equations of the other with respect to the parameters on replacing x_1 by $x_1 + \lambda$.

Let D_x represent the determinant of the finite equations of G with respect to the y's, so that

$$D_x = | \sum_i a_{sik} x_i | ,$$

the term written being the k^{th} element of the s^{th} row. If in D_x the variables be transformed by any operation of the group, then

$$D_{x'} = | \sum_i a_{sik} \sum_{j,l} a_{ijl} x_j y_l | .$$

Using in this the relation

$$\sum_i a_{sik} a_{ijl} = \sum_i a_{ilk} a_{sji},$$

$$D_{x'} = | \sum_i \sum_l a_{ilk} y_l \sum_j a_{sji} x_j |$$

$$= D_y D_x,$$

by the ordinary rule for the multiplication of determinants, D_y representing the result of writing y's for x's in D_x.

The determinant D_x is therefore a homogeneous function of the x's which is changed into a multiple of itself by every operation of G.

3. Let now P_x be any homogeneous irreducible function of the x's which has this property, so that

$$P_{x'} = f(y_1, y_2, \ldots, y_n) P_x.$$

In this equation put

$$x_1 = 1, \quad x_2 = x_3 = \ldots = x_n = 0,$$

so that, in virtue of the finite equations of the group,

$$x'_1 = y_1, \quad x'_2 = y_2, \quad \ldots, \quad x'_n = y_n.$$

Then $$P_y = f(y_1, y_2, \ldots, y_n) P_{1,0,\ldots}.$$

If the term of the highest degree in x_1 in P_x is wanting, this shows that P_x is zero. We may therefore assume that the coefficient of the term of the highest degree in x_1 is unity, and then

$$P_{x'} = P_y P_x.$$

Take $$x_1' = D_x, \quad x_2' = \ldots = x_n' = 0,$$

so that the y's are rational integral functions of the x's of degree $n-1$. If m is the degree of P_x, the equation becomes

$$D_x^m = P_y P_x,$$

and therefore P_x, being by supposition irreducible, must be a factor of D_x.

The relation $$P_{x'} = P_y P_x$$

may be regarded from the point of view of the reciprocal group; for, if in the equations

$$x_s' = \Sigma a_{sik} x_i y_k$$

we regard the y's as the original variables, the x's as the parameters, and the x''s as the transformed variables, the equations are in fact the finite equations of the reciprocal group; and $P_{x'}$ is the result of performing on the y's in P_y any operation of the reciprocal group. A homogeneous function of the variables which is transformed into a multiple of itself by every operation of G has therefore the same property in respect of G'.

Lastly, every factor of D_x has the property in question. For let P_x be any factor of D_x. Since

$$D_{x'} = D_y D_x,$$

it follows that $$P_{x'} = Q_y R_x,$$

where Q_y and R_x are factors of D_y and D_x respectively. In this equation put $$x_1 = 1, \quad x_2 = \ldots = x_n = 0,$$

so that $$x_r' = y_r \quad (r = 1, 2, \ldots, n).$$

Then $$P_y = Q_y R_{1,0,\ldots},$$

and therefore $$P_{x'} = P_y R_x \div R_{1,0,\ldots}.$$

Hence P_y is changed into a multiple of itself by every operation of the reciprocal group, and therefore

$$R_x \div R_{1,0,\ldots} = P_x \div P_{1,0,\ldots}.$$

If D_x' is the determinant of the finite equations of the reciprocal group with respect to the parameters, it follows that every irreducible function of the x's which has the property

$$P_{x'} = P_y P_x$$

is a factor of D_x', and that every factor of D_x' has this property. The two determinants D_x and D_x' therefore have the same irreducible factors; and they can only differ, if at all, in the powers in which *these factors enter into them.*

4. Every function P_x which has the property denoted by the equation

$$P_{x'} = P_y P_x,$$

in respect of the substitutions of G, is an invariant of the derived group of G. In fact, if X is one of the differential operations of G, the result of carrying out the corresponding infinitesimal operation on P_x is $(1+\delta t X) P_x$. For this infinitesimal operation, P_y differs from unity by a quantity proportional to δt, and therefore

$$XP_x = \lambda P_x,$$

where λ is a constant. Similarly, if Y is any other differential operator of G,

$$YP_x = \mu P_x,$$

where μ is a constant; and therefore

$$(XY) P_x = 0.$$

In particular, every factor of D_x is an invariant of the derived groups of both G and G'.

5. To avoid interrupting the argument later, it will be convenient here to deal with the case in which D_x has only one distinct invariant factor. In this case D_x and D'_x are identical, and the determinants of the finite equations of G or G' with respect to the variables are also identical, being given by writing y's for x's in D_x.

Suppose, first, that this irreducible factor is linear, so that

$$D_x = (x_1 + a_2 x_2 + \ldots + a_n x_n)^n.$$

The characteristic equation of the general finite operation of G is then

$$(\lambda + y_1 + a_2 y_2 + \ldots + a_n y_n)^n = 0.$$

Take, now, new variables, ξ's, and new parameters, η's, linear functions of the old ones, such that

$$\xi_1 = x_1 + a_2 x_2 + \ldots + a_n x_n,$$

$$\eta_1 = y_1 + a_2 y_2 + \ldots + a_n y_n.$$

The characteristic equation of the general operation of the group is then

$$(\lambda + \eta_1)^n = 0,$$

and the first of its finite equations is

$$\xi'_1 = \eta_1 \xi_1.$$

Moreover, ξ_1 is the only linear function of the variables which is transformed into a multiple of itself by every operation of the group. In the equations thus written put ξ_1 and ξ'_1 zero. The remaining $n-1$ equations (the first only disappearing) have the group property, and are the finite equations of a linear homogeneous group (not necessa[illegible]

transitive) in the $n-1$ variables in which the parameters enter linearly and homogeneously. Moreover, the characteristic equation of the general operation of this group is

$$(\lambda+\eta_1)^{n-1}=0.$$

If, in the finite equations of this group, $\eta_1\xi_r$ be written for ξ_r' for each suffix r from 2 to n, one (or more, if the group is intransitive) set of values of the ratios $\xi_2 : \xi_3 : \ldots : \xi_n$ is determined such that the set of values or point $(\xi_2, \xi_3, \ldots, \xi_n)$ is changed into $(\eta_1\xi_2, \eta_1\xi_3, \ldots, \eta_1\xi_n)$ by every one of its operations. Hence, by a suitable change of variables, one or more new variables can be chosen each of which is changed into η_1 times itself by every operation of this group. If ξ_2 be such a symbol, then for every operation of G the symbol replacing ξ_2 is of the form $\eta\xi_1+\eta_1\xi_2$, where η is a linear function of the parameters.

Repeating this process, it follows that the finite equations of G can be thrown into the form

$$\begin{aligned}
\xi_1' &= \eta_1\xi_1,\\
\xi_2' &= \eta_2\xi_1+\eta_1\xi_2,\\
\xi_3' &= \eta_3\xi_1+a\xi_2+\eta_1\xi_3,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,\\
\xi_n' &= \eta_n\xi_1+b\xi_2+c\xi_3+\ldots+\eta_1\xi_n.
\end{aligned}$$

Since the group is transitive, $\eta_1, \eta_2, \ldots, \eta_n$ must be linearly independent; and, since η_1 corresponds to the operation which changes every symbol in the same ratio, $a, b, c, \ldots$ are linear functions of $\eta_2, \eta_3, \ldots$. The group thus expressed is in standard form, and the finite equations of G' are therefore obtained by interchanging ξ's and η's on the right-hand side (*i.e.*, writing $\xi_p\eta_q$ for $\eta_p\xi_q$). Now, the first t equations of G form a transitive group in t variables, and the first t equations of G' form the reciprocal group. Hence each of these groups is simply transitive, and therefore no parameter with a greater suffix than t can occur in the equation for ξ_t'.

Moreover, in the expression for ξ_t' the only term in which η_t occurs is $\eta_t\xi_1$, since in the corresponding equation of G' the only term in which ξ_t occurs is $\eta_1\xi_t$. Hence in the group given by the first t equations of G the infinitesimal operation corresponding to η_t is self-conjugate.

Suppose, now, that the first $t-1$ equations of G taken alone constitute an *Abelian group*. Then, since G is its own parameter group,

the sub-group of G generated by the infinitesimal operations corresponding to $\eta_1, \eta_2, \ldots, \eta_{t-1}$ is Abelian. But, in the group constituted by the first t equations of G, the operation corresponding to η_t is self-conjugate. Hence this group is Abelian. The first two equations of G, however, clearly constitute an Abelian group; and therefore, by induction, G itself must be Abelian.

A simply transitive group, whose finite equations are linear and homogeneous in both the variables and the parameters, while its characteristic equation has only one distinct root, is therefore necessarily Abelian.

6. Suppose, secondly, that the irreducible factor of which D_x is a power is not linear, and let

$$P(x_1, x_2, \ldots, x_n)$$

represent it, where P is of the μ^{th} degree in the variables, and

$$D_x = P^\nu,$$

so that $$n = \mu\nu.$$

The characteristic equation of the general operation of G or G' is

$$\{P(y_1+\lambda, y_2, \ldots, y_n)\}^\nu = 1.$$

If $$P(y_1+\lambda, y_2, \ldots, y_n) \equiv (\lambda+\lambda_1)(\lambda+\lambda_2)\ldots(\lambda+\lambda_\mu),$$

so that $-\lambda_1, -\lambda_2, \ldots, -\lambda_\mu$ are the μ distinct roots of the characteristic equation, a general substitution of G' can, by properly choosing the variables, be thrown into the form

$$z'_{i1} = \lambda_i z_{i1}, \quad z'_{i2} = \lambda_{i2} z_{i2} + \lambda_i z_{i2}, \quad \ldots, \quad z'_{i\nu} = \lambda_{i,\nu-1} z_{i,\nu-1} + \lambda_i z_{i\nu}$$
$$(i = 1, 2, \ldots, \mu).$$

Every substitution on the $\mu\nu$ variables which is permutable with this, and therefore every substitution of G, must transform among themselves each set of z's with the same first suffix. Hence the characteristic determinant of G in its new form consists of the product of μ distinct factors $f_1(\lambda), f_2(\lambda), \ldots, f_\mu(\lambda)$, rational in λ and the parameters, each of which is the characteristic determinant of the group so far as it affects a single set of z's. The characteristic determinant of G being invariant, we have

$$f_1(\lambda) f_2(\lambda) \ldots f_\mu(\lambda) = \left[P(y_1+\lambda, y_2, \ldots, y_n)\right]^\nu.$$

Hence, since P is irreducible, $\mu = \nu$,

and $$f_1(\lambda) = f_2(\lambda) = \ldots = P(y_1+\lambda, y_2, \ldots, y_n).$$

This second case can therefore only occur when n is a square.

Suppose, next, that any operation of G leaves all the z's of one set unchanged. For such an operation $P(y_1+\lambda, y_2, \ldots, y_n)$ must be $(\lambda+1)^r$, and its first invariant factor must be $\lambda+1$; and then each of the z's is unchanged. Hence G is simply isomorphic with the group which arises when we consider the effect of the substitutions of G on one set of the z's only. This is a group of order μ^2, linear and homogeneous in μ variables. There is but one such group, namely the general linear homogeneous group in μ variables.

Hence, when the determinant of the finite equations of a simply transitive group in n variables is the power of a single irreducible non-linear factor, n must be a square, and the group is simply isomorphic with the general linear homogeneous group in $\sqrt{n}$ variables.

Further, it is clear from this discussion that, in the case under consideration, the finite equations of the group G may be written in the form

$$\left.\begin{aligned} z'_{i1} &= u_{11}z_{i1}+u_{12}z_{i2}+\ldots+u_{1\mu}z_{i\mu} \\ z'_{i2} &= u_{21}z_{i1}+u_{22}z_{i2}+\ldots+u_{2\mu}z_{i\mu} \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ z'_{i\mu} &= u_{\mu 1}z_{i1}+u_{\mu 2}z_{i2}+\ldots+u_{\mu\mu}z_{i\mu} \end{aligned}\right\} \quad (i=1, 2, \ldots, \mu),$$

the u's being the parameters and n being equal to μ^2. The finite equations of G' in terms of the same variables are

$$\left.\begin{aligned} z'_{1i} &= u_{11}z_{1i}+u_{12}z_{2i}+\ldots+u_{1\mu}z_{\mu i} \\ z'_{2i} &= u_{21}z_{1i}+u_{22}z_{2i}+\ldots+u_{2\mu}z_{\mu i} \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ z'_{\mu i} &= u_{\mu 1}z_{1i}+u_{\mu 2}z_{2i}+\ldots+u_{\mu\mu}z_{\mu i} \end{aligned}\right\} \quad (i=1, 2, \ldots, \mu).$$

The determinant of the finite equations of G with respect to the parameters is the μ^{th} power of

$$\begin{vmatrix} z_{11}, & z_{12}, & \ldots, & z_{1\mu} \\ z_{21}, & z_{22}, & \ldots, & z_{2\mu} \\ \ldots & \ldots & \ldots & \\ z_{\mu 1}, & z_{\mu 2}, & \ldots, & z_{\mu\mu} \end{vmatrix},$$

and this is changed into a multiple of itself, as it should be, whether either the variables in every line or the variables in every column are transformed among themselves by any the same linear homogeneous substitution.

II.

7. It will be convenient to modify slightly the notation used in my former paper, so that the continuous group G, which is connected with a given group g of finite order n, shall be in what I have called "standard form."

With this end in view, the finite equations of G are taken in the form

$$x'_s = \sum_{i=1}^{i=n} y_{si^{-1}} x_i \quad (s = 1, 2, \ldots, n),$$

where

$$y_{si^{-1}} = y_t,$$

if

$$S_s S_i^{-1} = S_t;$$

S_s, S_i, S_t being three of the operations of g.

If G' is the reciprocal group, its finite equations are

$$x'_s = \sum_{i=1}^{i=n} y_{i^{-1}s} x_i \quad (s = 1, 2, \ldots, n);$$

and every operation of G' is permutable with every operation of G. Moreover, every linear homogeneous substitution on the n symbols which is permutable with every operation of G is contained in G', and conversely. The sub-group H of G constituted by the totality of its self-conjugate operations is the greatest common sub-group of G and G'. It is shown in my former paper (pp. 213, 221) that the order of H is r, where r is the number of distinct conjugate sets of operations in g; and that the finite equations of H are

$$x'_s = \sum_{i=1}^{i=n} e_{si^{-1}} x_i \quad (s = 1, 2, \ldots, n),$$

where the double suffixes of the e's are reduced by the same law as those of the y's, and

$$e_i = e_j,$$

if S_i and S_j are conjugate operations of g.

Of the e's only r are independent. The r independent symbols will be taken to be

$$\epsilon_1, \epsilon_2, \ldots, \epsilon_r;$$

and, if S_i belongs to the k^{th} conjugate set of operations in g, when these sets are arranged in some definite sequence to which the sequence of the ϵ's corresponds, then

$$e_i = \epsilon_k.$$

The first conjugate set will be taken to be that consisting of the identical operation alone, so that

$$e_1 = \epsilon_1.$$

8. The characteristic equation of the general finite operation of H is

$$\begin{vmatrix} e_1+\lambda, & e_{2^{-1}}, & e_{3^{-1}}, & \ldots, & e_{n^{-1}} \\ e_2, & e_1+\lambda, & e_{23^{-1}}, & \ldots, & e_{2n^{-1}} \\ e_3, & e_{32^{-1}}, & e_1+\lambda, & \ldots, & e_{3n^{-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_n, & e_{n2^{-1}}, & e_{n3^{-1}}, & \ldots, & e_1+\lambda \end{vmatrix} = 0.$$

The determinant on the left-hand side of this equation will be denoted by $D_{e,\lambda}$, and the corresponding determinant when λ is zero by D_e; so that $D_{e,\lambda}$ is the result of writing $e_1+\lambda$ for e_1 in D_e.

It is shown in my paper "On Linear Continuous Homogeneous Groups whose Operations are Permutable"* that the characteristic determinant of the general finite operation of an Abelian group which is linear and homogeneous both in the variables and the parameters is the product of factors which are linear and homogeneous in λ and the parameters. We may therefore write

$$D_{e,\lambda} = \prod_{i=1}^{i=s} (\lambda+\epsilon_1+a_{i2}\epsilon_2+\ldots+a_{ir}\epsilon_r)^{m_i},$$

where

$$\sum_{i=1}^{i=s} m_i = n.$$

If n_k is the number of operations in the k^{th} conjugate set of g, the symbol ϵ_k will occur n_k times in each line of D_e, and, therefore, nn_k times in all. Now it may be shown, and the result is most important for what follows, that the minor corresponding to ϵ_k is in each place the same. To prove this, let T be the operation of H whose parameters

are

$$e_1, e_2, \ldots, e_n;$$

and let

$$\eta_1, \eta_2, \ldots, \eta_n$$

be the parameters of T^{-1}.

Since $T^{-1}T$ is the identical operation, the e's and η's are connected by the relations

$$\sum_{s=1}^{s=n} \eta_{si^{-1}} e_{is^{-1}} = 1,$$

$$\sum_{s=1}^{s=n} \eta_{si^{-1}} e_{js^{-1}} = 0 \quad (i \neq j),$$

for all values of i and j.

* *Proc. Lond. Math. Soc.*, Vol. XXIX., p. 332; where other references are given.

Hence $\eta_{si^{-1}}$ is the minor of $e_{is^{-1}}$ in the determinant

$$\begin{vmatrix} e_{11^{-1}}, & e_{12^{-1}}, & e_{13^{-1}}, & \dots, & e_{1n^{-1}} \\ e_{21^{-1}}, & e_{22^{-1}}, & e_{23^{-1}}, & \dots, & e_{2n^{-1}} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ e_{n1^{-1}}, & e_{n2^{-1}}, & e_{n3^{-1}}, & \dots, & e_{nn^{-1}} \end{vmatrix},$$

where the element $e_{is^{-1}}$ is to be picked out before the double suffixes are simplified; and, after simplifying, the e's may be replaced by the equivalent ϵ's. But, T^{-1} being an operation of H, $\eta_{si^{-1}}$ depends only on the conjugate set in g to which $S_s S_i^{-1}$ belongs. Hence, in the above determinant, which is D_ϵ, the minors of $e_{is^{-1}}$ and $e_{jt^{-1}}$ are the same if $S_i S_s^{-1}$ and $S_j S_t^{-1}$ belong to the same conjugate set in g. In other words, when in D_ϵ the e's are replaced by ϵ's, the minor of ϵ_k is the same in every place in which it occurs. Let M_k denote the minor of ϵ_k. Then, by differentiating the determinant,

$$M_k = \frac{1}{nn_k} \frac{\partial D_\epsilon}{\partial \epsilon_k}.$$

Hence, m_i being the order to which any linear factor

$$\lambda + \epsilon_1 + a_{i2}\epsilon_2 + \dots + a_{ir}\epsilon_r$$

enters in $D_{\epsilon,\lambda}$, the same factor enters to the order $m_i - 1$ in every first minor of $D_{\epsilon,\lambda}$. The first invariant-factor of $D_{\epsilon,\lambda}$ therefore contains every linear factor of the determinant, and each one in the first degree. I have shown (*loc. cit.*, p. 347) that, where this is the case, n independent linear functions of the variables can be chosen so that each of them is changed into a multiple of itself by every operation of the group; the multipliers being the roots of the characteristic equation taken negatively, each occurring as often as the corresponding root. The order of such a group is obviously equal to the number of distinct multipliers. Now the order of H is r. Hence $s = r$, and the equation

$$D_{\epsilon,\lambda} = 0$$

has just r distinct roots. Let these roots be

$$-\eta_1, \ -\eta_2, \ \dots, \ -\eta_r,$$

so that $$D_{\epsilon,\lambda} = (\lambda + \eta_1)^{m_1} (\lambda + \eta_2)^{m_2} \dots (\lambda + \eta_r)^{m_r},$$

where $$\sum_{i=1}^{i=r} m_i = n.$$

Then the result of this discussion is that n independent linear functions of the x's

$$\xi_{ij} \quad (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m_i),$$

may be chosen so that the finite equations of H take the form

$$\xi'_{ij} = \eta_i \xi_{ij} \quad (j = 1, 2, \ldots, m_i;\ i = 1, 2, \ldots, r).$$

9. Suppose, next, that the finite equations of G and G' are expressed in terms of the ξ's. Since every operation of either G or G' is permutable with every operation of H, every operation of G and of G' must transform among themselves each set of ξ's with the same first suffix. Let G_i and G'_i be the groups to which G and G' reduce when we consider only their effect on the set of m_i symbols

$$\xi_{ij} \quad (j = 1, 2, \ldots, m_i).$$

Since the finite equations of G and G' contain the parameters linearly and homogeneously, the same must be true of the finite equations of G_i and G'_i; and, since G and G' are transitive, G_i and G'_i must be transitive in the m_i symbols on which they operate. Also, since every operation of G is permutable with every operation of G', the same is true of G_i and G'_i. Hence G_i and G'_i must be simply transitive in the m_i symbols on which they operate; in other words, they are each of order m_i. Since G_i is a simply transitive group in m_i variables whose finite equations are linear and homogeneous both in the variables and parameters, these finite equations can be written in the form

$$\xi'_{is} = \sum_{j,k}^{1, \ldots, m_i} \alpha_{sjk} \xi_{ij} \eta_{ik} \quad (s = 1, 2, \ldots, m_i),$$

where η_{ik} $(k = 1, 2, \ldots, m_i)$, the m_i independent parameters, are independent linear functions of the original y's. Moreover, by carrying out, if necessary, a linear substitution on the ξ's, which leaves the form of H unaltered, these equations may be brought to standard form; and the finite equations of G'_i are then

$$\xi'_{is} = \sum_{j,k}^{1, \ldots, m_i} \alpha_{skj} \xi_{ij} \eta_{ik} \quad (s = 1, 2, \ldots, m_i).$$

Consider, now, the n linear functions

$$\eta_{ij} \quad (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m_i)$$

of the original parameters thus introduced. If they are not independent, the finite equations of G', when expressed in terms of the ξ's,

would contain fewer than n independent parameters. This is impossible, since G is a group of order n. Hence the n new parameters are independent, and therefore the group G_i, to which G reduces when we consider only its effect on the symbols ξ_{ik} $(k = 1, 2, \ldots, m_i)$, is a sub-group of G. Moreover, this sub-group and the sub-group G_j, which affects only the symbols ξ_{jk} $(k = 1, 2, \ldots, m_j)$, have no operation in common except identity; and, since the symbols affected by the two sub-groups are distinct, every operation of one is permutable with every operation of the other. It follows, then, that G is the direct product of

$$G_1, \; G_2, \; \ldots, \; G_r.$$

For each suffix i, the self-conjugate operations of G_i form a group of order 1. In fact, the self-conjugate operations of G_i are also self-conjugate operations of G; hence, if the self-conjugate operations of G_i formed a sub-group whose order were greater than 1, the self-conjugate operations of G would form a sub-group of order greater than r.

...

10. Consider, next, the determinant of the finite equations of G_i with respect to the parameters. It, and each of its irreducible factors, is an invariant of the derived group of G_i, and, therefore, also of the derived group of G. The r determinants of the r groups G_i $(i = 1, 2, \ldots, r)$ therefore give r invariants of the derived group of G; and, since the variables that enter into any one of these are distinct from those which enter into any other, these r invariants are independent. Now the derived group of G, being a sub-group of order $n-r$ of a transitive group of order n in n variables, cannot have more than r independent invariants. Hence the determinant of the finite equations of G_i, with respect to the parameters, must be the power of a single irreducible factor for every value of i. For, if it contained two or more irreducible factors, each of them would be invariants of the derived group of G, and at least two of them would be independent; this would, however, make the number of independent invariants of the derived group of G greater than r. Now the self-conjugate operations of G_i form a group of order 1. Hence, if m_i is greater than 1, the irreducible factor in question cannot be linear (§ 5). It follows therefore (§ 6) that, if m_i is greater than unity, it must be a square, and G_i is simply isomorphic with the general linear homogeneous group in $\sqrt{m_i}$ variables.

As regards the continuous group G of order n in n variables, which is defined by the group g of finite order n containing r distinct sets of

conjugate operations, the result of the present section may be summarized thus:—

The continuous group G is the direct product of r groups simply isomorphic with the general linear homogeneous groups in $\mu_1 (= 1)$, $\mu_2, \ldots, \mu_r$ variables, where the integers $\mu_1, \mu_2, \ldots$ are such that $\sum_1^r \mu_i^2 = n$.*

As regards the group g of finite order n, the results may also be stated as follows:—

The determinant formed from the multiplication table of g, when the symbols of the operations are regarded as n independent algebraical quantities, contains r distinct irreducible factors. Each of these factors enters into the determinant to a power equal to its degree†; and any irreducible factor of degree m is expressible in the form

$$\begin{vmatrix} z_{11}, & z_{12}, & \ldots, & z_{1m} \\ z_{21}, & z_{22}, & \ldots, & z_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ z_{m1}, & z_{m2}, & \ldots, & z_{mm} \end{vmatrix},$$

where the z's are independent linear functions of the n algebraical quantities.

11. Let P_x be an irreducible factor of D_x whose degree is μ_i. Then it has been seen (§ 6) that μ_i^2 independent linear functions of the x's can be formed such that

$$P_x = \begin{vmatrix} \xi^i_{11}, & \xi^i_{12}, & \ldots, & \xi^i_{1\mu_i} \\ \xi^i_{21}, & \xi^i_{22}, & \ldots, & \xi^i_{2\mu_i} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \xi^i_{\mu_i 1}, & \xi^i_{\mu_i 2}, & \ldots, & \xi^i_{\mu_i \mu_i} \end{vmatrix},$$

while the functions in each line (or column) of the determinant are transformed among themselves by every operation of G (or G').

Now, $$P_{x'} = P_y P_x;$$

hence, if η^i_{pq} $(p, q = 1, 2, \ldots, \mu_i)$ denote the same μ_i^2 linear functions

* That one at least of the μ's must be unity follows from noticing that the determinant D_y obviously contains the linear factor $y_1 + y_2 + \ldots + y_n$.

† *Frobenius: Berliner Sitzungsberichte*, 1896, pp. 1363, 1372.

of the y's that the ξ^i's are of the x's, the η^i's may be taken as the independent parameters of G_i, and the finite equations of G_i take the form

$$\left.\begin{aligned}
\xi'^i_{s1} &= \eta^i_{11}\,\xi^i_{s1} + \eta^i_{12}\,\xi^i_{s2} + \ldots + \eta^i_{1\mu_i}\,\xi^i_{s\mu_i} \\
\xi'^i_{s2} &= \eta^i_{21}\,\xi^i_{s1} + \eta^i_{22}\,\xi^i_{s2} + \ldots + \eta^i_{2\mu_i}\,\xi^i_{s\mu_i} \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\xi'^i_{s\mu_i} &= \eta^i_{\mu_i 1}\xi^i_{s1} + \eta^i_{\mu_i 2}\xi^i_{s2} + \ldots + \eta^i_{\mu_i\mu_i}\xi^i_{s\mu_i}
\end{aligned}\right\} \quad = 1, 2, \ldots, \mu_i).$$

When the y's are replaced by e's, these equations reduce to

$$\xi'^i_{pq} = \eta_i \xi^i_{pq} \quad (p, q = 1, 2, \ldots \mu_i);$$

and therefore the η^i's are such that, on replacing y's by e's,

$$\eta^i_{pq} = 0 \quad (p \neq q),$$

$$\eta^i_{pp} = \eta_i.$$

The n operations of G given by

$$y_1 = \ldots = y_{k-1} = y_{k+1} = \ldots = y_n = 0,\ y_k = 1 \quad (k = 1, 2, \ldots, n)$$

constitute a discontinuous sub-group of finite order n simply isomorphic with g. Hence the distinct operations of G_i which correspond to these values of the parameters constitute a discontinuous sub-group g_i of G_i with which g is isomorphic. If the isomorphism is not simple, let

$$S_1,\ S_2,\ \ldots,\ S_m$$

constitute the self-conjugate sub-group of g which corresponds to the identical operation of g_i. Then

$$S_t S_1,\ S_t S_2,\ \ldots,\ S_t S_m$$

correspond to the same operation of g_i, and therefore in the η^i's the symbols

$$y_{t1},\ y_{t2},\ \ldots,\ y_{tm}$$

occur only in the linear combination

$$y_{t1} + y_{t2} + \ldots + y_{tm}$$

for each value of t. The η^i's are therefore linear functions of $\frac{n}{m}$ linear functions of the y's; and, since they are independent and μ_i^2 in number,

$$\frac{n}{m} \geqslant \mu_i^2,$$

i.e.,

$$m \leqslant \frac{n}{\mu_i^2}.$$

Hence, unless g has a self-conjugate sub-group whose index is equal to or greater than μ_i^2, g_i must be simply isomorphic with g. In particular, if g is simple, g_i is simply isomorphic with g, except when μ_i is unity; in this case g_i clearly consists of the identical operation only. Again, when g contains a simple self-conjugate sub-group of index 2, g_i and g are simply isomorphic unless μ_i is unity.

Now g, and the groups of smaller order with which g is isomorphic, can, in general, be represented as groups of linear substitutions in a variety of ways, and the question arises as to which of these different modes of representation occur among the groups g_i. This question is best considered by approaching it from a rather different point of view.

12. Let
$$S_1 (=1),\ S_2,\ \ldots,\ S_\nu$$
be the operations of a group g' of finite order ν, which can be represented as a group of linear substitutions performed on the m symbols
$$x_1,\ x_2,\ \ldots,\ x_m;$$
and let
$$x_s' = \sum_{i=1}^{i=m} a_{sik} x_i \quad (s = 1, 2, \ldots, m)$$
be the substitution corresponding to the operation S_k of g'.

Then, if
$$S_k S_i = S_p,$$
$$\sum_{j=1}^{j=m} a_{sji} a_{jik} = a_{sip}$$
for all values of s and i.

Let Y and Z represent the two linear substitutions
$$\left.\begin{aligned} x_s' &= \sum_{i,k} a_{sik} x_i y_k \\ \text{and} \quad x_s' &= \sum_{i,k} a_{sik} x_i z_k \end{aligned}\right\} \quad \begin{pmatrix} s, i = 1, 2, \ldots, m \\ k = 1, 2, \ldots, \nu \end{pmatrix};$$
where
$$y_1,\ y_2,\ \ldots,\ y_\nu,$$
$$z_1,\ z_2,\ \ldots,\ z_\nu,$$
are any two sets of constants.

The linear substitution YZ is given by the equations
$$\begin{aligned} x_s' &= \sum_{j,l} a_{sjl} y_l \sum_{i,k} a_{jik} x_i z_k \\ &= \sum_{i,l,k} x_i y_l z_k \sum_j a_{sjl} a_{jik} \\ &= \sum_{i,l,k} a_{sip} x_i y_l z_k, \end{aligned}$$

where the suffix p is determined by the relation

$$S_k S_l = S_p.$$

If, then, we write $$w_p = \Sigma y_l z_k,$$

where the summation is extended to all pairs of suffixes such that

$$S_k S_l = S_p,$$

the linear substitution YZ is given by the equations

$$x'_s = \Sigma a_{sip} x_i w_p.$$

The relation of the w's to the y's and z's may be more conveniently represented by means of the notation with multiple suffixes already used, such that

$$y_{ab\ldots d} = y_e$$

and $$z_{ab\ldots d} = z_e,$$

if $$S_a S_b \ldots S_d = S_e.$$

With this notation the w's are given in terms of the y's and z's by the equations

$$w_p = \sum_{l=1}^{l=\nu} z_{pl^{-1}} y_l = \sum_{k=1}^{k=\nu} z_k y_{k^{-1}p} \quad (p = 1, 2, \ldots, \nu).$$

These are the finite equations of the continuous group G' of order ν in ν variables, defined by the multiplication table of the group g'. The totality of the linear substitutions of the form Y, when the y's are regarded as arbitrary parameters, form therefore a continuous group Γ', with which G' is isomorphic.

The isomorphism between G' and Γ' is clearly not necessarily, nor in general, simple. In fact, the order of Γ', a linear homogeneous group in m variables, cannot exceed m^2, and there is no such limitation on ν.

13. Suppose, next, that g is a continuous group of order $n\,(=\mu\nu)$, and that g contains a self-conjugate sub-group h of order μ, in respect of which it is isomorphic with g'. If

$$S_{\alpha_1}(=1),\ S_{\alpha_2},\ \ldots,\ S_{\alpha_\mu}$$

are the operations of h, a set of ν operations

$$S_1(=1),\ S_2,\ \ldots,\ S_\nu$$

may be chosen so that the totality of the operations of g are represented by the ν sets

$$S_t S_{\alpha_1},\ S_t S_{\alpha_2},\ \ldots,\ S_t S_{\alpha_\mu} \quad (t = 1, 2, \ldots, \nu),$$

while these sets combine among themselves in the same way as the operations of g'.

Using the same suffix notation as before, consider the linear substitution Y, or

$$x'_s = \Sigma \beta_{si(ta_r)} x_i y_{ta_r} \quad \begin{pmatrix} s, i = 1, 2, \ldots, m \\ t = 1, 2, \ldots, \nu \\ r = 1, 2, \ldots, \mu \end{pmatrix},$$

where $$\beta_{si(ta_r)} = \alpha_{sit}.$$

It contains, formally, n arbitrary parameters corresponding to the n operations of g. If the double suffixes ta_r are replaced by single suffixes extending from 1 to n, Y may be written in the form

$$x'_s = \Sigma \beta_{sik} x_i y_k \quad \begin{pmatrix} s, i = 1, 2, \ldots, m \\ k = 1, 2, \ldots, n \end{pmatrix},$$

and, exactly as in the former case, it may be verified that YZ is

$$x'_s = \Sigma \beta_{sik} x_i w_k,$$

where $$w_k = \sum_{l=1}^{l=n} z_{kl^{-1}} y_l \quad (k = 1, 2, \ldots, n).$$

The set of equations $$x'_s = \Sigma \beta_{sik} x_i y_k$$

therefore define a continuous group Γ with which the continuous group G, of order n, formed from the multiplication table of g, is isomorphic.

Let $$\Delta_y = | \sum_k \beta_{sik} y_k |$$

denote the determinant of the general finite operation of Γ. The only inadmissible systems of values of the parameters are those given by equating Δ_y to zero. But, since G is isomorphic with Γ, these systems of values must all occur among those obtained by equating D_y to zero. The irreducible factors of Δ_y are therefore factors of D_y, and we may write

$$\Delta_y = P_{a_1}^{\nu_1} P_{a_2}^{\nu_2} \ldots P_{a_t}^{\nu_t},$$

where $$P_{a_1}, P_{a_2}, \ldots, P$$

are irreducible factors of D_y of degrees

$$\mu_{a_1}, \mu_{a_2}, \ldots, \mu_a,$$

which correspond to the self-conjugate sub-groups of G,

$$G_{a_1}, G_{a_2}, \ldots, G_{a_t},$$

whose orders are $\mu_{a_1}^2, \mu_{a_2}^2, \ldots, \mu_{a_t}^2$.

If P_i is an irreducible factor of D_y which does not occur in Δ_y, and G_i the corresponding self-conjugate sub-group of G of order μ_i^2, then to every operation of G_i must correspond the identical operation of Γ. On the other hand, no operation of G_{a_i}, except the identical operation, can correspond to the identical operation of Γ; for the characteristic determinant of the general finite operation of Γ contains that of G_{a_i} as a factor. Hence Γ is simply isomorphic with the direct product of $G_{a_1}, G_{a_2}, \ldots, G_{a_t}$.

The characteristic equation of the most general self-conjugate operation of Γ will then have t distinct roots, and therefore the m variables of Γ, or suitably chosen linear functions of them which may be taken as new variables, are capable of being divided into t sets, such that the variables of each set are transformed among themselves by all the operations of Γ.

The numbers of variables in the t sets are clearly

$$\nu_1\mu_{a_1}, \nu_2\mu_{a_2}, \ldots, \nu_t\mu_{a_t}.$$

Further, since the finite equations of Γ are linear in the parameters, the $\nu_s\mu_{a_s}$ symbols of the s^{th} set may be replaced by ν_s new sets, of μ_{a_s} symbols each, such that the symbols of each set are transformed among themselves by all the operations of Γ.

This completes the analysis of the group Γ, and affords a partial answer to the question suggested at the end of § 11, which may be stated thus :—If a discontinuous group g', with which g is simply or multiply isomorphic, can be represented as a group of linear substitutions performed on m symbols, and if it is impossible, by choosing new variables, to divide the variables into sets such that those of each set are transformed among themselves by every operation of g', then m must be one of the integers μ_s defined in § 10.

On the General Theory of Anharmonics. *By* Prof. EDGAR O. LOVETT.

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It is the object of this note to show how Lie's theory of the projective group may be made to yield Clifford's statement* of the general theory of anharmonics.

A geometrical configuration depending on $m \,.\, n$ coordinates, say a system of m points in a space of n dimensions, has at least $mn-r$ absolute invariants under an r-parameter Lie group of point transformations; these $mn-r$ invariant functions, and all others, are solutions of the complete system of simultaneous partial differential equations

$$\sum_1^m i(X_1 f)_i = 0, \quad \sum_1^m i(X_2 f)_i = 0, \quad \ldots, \quad \sum_1^m i(X_r f)_i = 0, \tag{1}$$

where the

$$X_k f = \sum_1^n i\, \xi_{ik}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} \quad (k = 1, \ldots, r) \tag{2}$$

are the r independent infinitesimal point transformations which generate the r-parameter continuous group, and $(X_k f)_j$ represents the result of replacing $x_1, \ldots, x_n$ by $x_1^{(j)}, \ldots, x_n^{(j)}$, respectively, in $(X_k f)$,

$$x_1^{(j)}, \; x_2^{(j)}, \; \ldots, \; x_n^{(j)} \quad (j = 1, 2, \ldots, m) \tag{3}$$

being the mn independent coordinates which determine the system of points.

1. Consider the general projective group of the xy-plane, an 8-parameter Lie group generated by the following independent infinitesimal point transformations—

$$\frac{\partial f}{\partial x}, \; \frac{\partial f}{\partial y}, \; x\frac{\partial f}{\partial y}, \; y\frac{\partial f}{\partial x}, \; x\frac{\partial f}{\partial x}, \; y\frac{\partial f}{\partial y}, \; x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y}, \; xy\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y}. \tag{4}$$

* Clifford: "On the General Theory of Anharmonics," *Proc. Lond. Math. Soc.*, Vol. II.

A plane hexagon regarded as a system of six points depending upon the twelve coordinates, $x_1, y_1, \ldots, x_6, y_6$, possesses at least four independent invariants under the above group. These and all others are found by integrating the complete system of partial differential equations

$$\left.\begin{aligned} &\sum_1^6 i\frac{\partial\phi}{\partial x_i} = 0, \quad \sum_1^6 i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i x_i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i y_i\frac{\partial\phi}{\partial x_i} = 0, \\ &\sum_1^6 i x_i\frac{\partial\phi}{\partial x_i} = 0, \quad \sum_1^6 i y_i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i\left(x_i^2\frac{\partial\phi}{\partial x_i} + x_i y_i\frac{\partial\phi}{\partial y_i}\right) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \sum_1^6 i\left(x_i y_i\frac{\partial\phi}{\partial x_i} + y_i^2\frac{\partial\phi}{\partial y_i}\right) = 0 \end{aligned}\right\}. \quad (5)$$

That this system of partial differential equations possesses no more than four independent solutions is readily seen by constructing the matrix

$$\left\|\begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & y_1^2 & y_2^2 & y_3^2 & y_4^2 \end{matrix}\right\}, \quad (6)$$

formed by the coefficients of the $\frac{\partial\phi}{\partial x_i}$ and $\frac{\partial\phi}{\partial y_i}$, and observing that all eighth order determinants of this matrix do not vanish; for example, the determinants composed of the first eight columns.

The first and second equations of the system (5) show that the

invariant function, ϕ, is a function of the variables

$$u_{i-1} = x_i - x_1, \quad v_i = y_i - y_1 \quad (i = 2, 3, \ldots, 6). \quad (7)$$

The introduction of these new variables in the third, fourth, fifth, and sixth equations of the system reduces them to the forms, respectively,

$$\sum_1^5 i u_i \frac{\partial \phi}{\partial v_i} = 0, \quad \sum_1^5 i v_i \frac{\partial \phi}{\partial u_i} = 0, \quad \sum_1^5 i u_i \frac{\partial \phi}{\partial u_i} = 0, \quad \sum_1^5 i v_i \frac{\partial \phi}{\partial v_i} = 0. \quad (8)$$

The first and second of these equations demand that ϕ shall be a function of the determinants

$$w_{ij} = | u_i, v_j | , \quad (9)$$

where ij takes successively the values of the ten combinations of the five integers 1, 2, 3, 4, 5, taken two at a time. On introducing the variables (9) in the third and fourth equations of (8), we find the form

$$\Sigma w_{ij} \frac{\partial \phi}{\partial w_{ij}} = 0, \quad (10)$$

which requires that the function ϕ be a function of the ratios of the determinants w_{ij}. Finally, by direct substitution in the transformed forms of the seventh and eighth equations of the system (5), we have the result that the following nine functions are solutions of the system, and hence invariant under the group (4),

$$\frac{w_{\alpha\beta}(w_{\lambda\mu} + w_{\mu\nu} + w_{\nu\lambda})}{w_{ab}(w_{lm} + w_{mn} + w_{nl})}, \quad (11)$$

where $$(a, b, l, m, n) = (1, 2, 3, 4, 5),$$

and $(\alpha\beta\lambda\mu\nu)$ is any other of the permutations of the five integers 1, 2, 3, 4, 5.

These invariants are readily interpreted geometrically; thus, for example, the expression

$$\frac{w_{13}(w_{24} + w_{45} + w_{52})}{w_{ab}(w_{lm} + w_{mn} + w_{nl})}, \quad (12)$$

when simplified, becomes successively

$$\frac{|\,u_1, v_3\,|\,\{\,|\,u_2, v_4\,|+|\,u_4, v_5\,|+|\,u_5, v_2\,|\,\}}{|\,u_1, v_2\,|\,\{\,|\,u_3, v_4\,|+|\,u_4, v_5\,|+|\,u_5, v_3\,|\,\}}$$

$$=\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_4-x_1 & y_4-y_1 \end{vmatrix}\cdot\begin{vmatrix} x_3-x_1 & y_3-y_1 & 1 \\ x_5-x_1 & y_5-y_1 & 1 \\ x_6-x_1 & y_6-y_1 & 1 \end{vmatrix}\Bigg/\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}\cdot\begin{vmatrix} x_4-x_1 & y_4-y_1 & 1 \\ x_5-x_1 & y_5-y_1 & 1 \\ x_6-x_1 & y_6-y_1 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\cdot\begin{vmatrix} x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \end{vmatrix}\Bigg/\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\cdot\begin{vmatrix} x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \end{vmatrix}$$

$$=\frac{124.356}{123.456}, \tag{13}$$

where 123 means the area of the triangle formed by the points 1, 2, 3. The other invariants are composed of corresponding terms formed from the twenty triangles made by the original six points. These nine invariants are not independent, since they may be shown to satisfy five independent identities of the form

$$123.456-124.563+125.634-126.345\equiv 0. \tag{14}$$

To prove this statement, we first establish the identity (14) by observing that, since $ijk=-kji$, the left-hand member is identical with the expression

$$123.456+124.365+125.634+126.543, \tag{15}$$

which, in terms of the quantities w_{ij}, in virtue of (12) and (13), may be written

$$\begin{aligned} &w_{12}(w_{34}+w_{45}+w_{53})+w_{13}(w_{25}+w_{54}+w_{42}) \\ &+w_{14}(w_{23}+w_{35}+w_{52})+w_{15}(w_{43}+w_{32}+w_{24}), \end{aligned} \tag{16}$$

or

$$\begin{aligned} &(w_{12}w_{34}+w_{23}w_{14}+w_{31}w_{24})+(w_{12}w_{45}+w_{24}w_{15}+w_{41}w_{25}) \\ &+(w_{12}w_{53}+w_{25}w_{13}+w_{51}w_{23})+(w_{13}w_{54}+w_{35}w_{14}+w_{51}w_{34}), \end{aligned} \tag{17}$$

which, by noting the definition (9) of the w_{ij}'s, is zero, by virtue of the identity

$$| a_1, b_2 | . | a_3, b_4 | + | a_2, b_3 | . | a_1, b_4 | + | a_3, b_1 | . | a_2, b_4 | \equiv 0. \tag{18}$$

In the second place, to determine the number of independent identities of the form (14), we remark that the demonstration of (14) just given demonstrates simultaneously the following system of identities derived by permutation:

$$\left.\begin{array}{l}
\left.\begin{array}{l} 123.456-124.563+125.634-126.345=0, \\ 124.563-125.634+126.345-123.456=0, \\ 125.634-126.345+123.456-124.563=0, \\ 126.345-123.456+124.563-125.634=0; \end{array}\right\} \\
\left.\begin{array}{l} 134.562-135.624+136.245-132.456=0, \\ 135.624-136.245+132.456-134.562=0, \\ 136.245-132.456+134.562-135.624=0; \end{array}\right\} \\
\left.\begin{array}{l} 145.623-146.235+142.356-143.562=0, \\ 146.235-142.356+143.562-145.623=0; \end{array}\right\} \\
156.234-152.346+153.462-154.623=0; \\
\left.\begin{array}{l} 234.561-235.614+236.145-231.456=0, \\ 235.614-236.145+231.456-234.561=0, \\ 236.145-231.456+234.561-235.614=0; \end{array}\right\} \\
\left.\begin{array}{l} 245.613-246.135+241.356-243.561=0, \\ 246.135-241.356+243.561-245.613=0; \end{array}\right\} \\
256.134-251.346+253.461-254.613=0; \\
\left.\begin{array}{l} 345.612-346.125+341.256-342.561=0, \\ 346.125-341.256+342.561-345.612=0; \end{array}\right\} \\
356.124-351.246+352.461-354.612=0; \\
456.123-451.236+452.361-453.612=0
\end{array}\right\}. \tag{19}$$

By designating the first ten terms in the left-hand column from the top down or the bottom up by $a, b, c, \ldots, j$, respectively, and observing

that the first four equations are identical, the next three, and so on, the system (19) reduces to the following system of ten identities:

$$a-b+c-d=0, \tag{20}$$
$$a-e+f-g=0, \tag{21}$$
$$b-e+h-i=0, \tag{22}$$
$$c-f+h-j=0, \tag{23}$$
$$a+h-i+j=0, \tag{24}$$
$$b+f-g+j=0, \tag{25}$$
$$c+e-g+i=0, \tag{26}$$
$$c-d+e+j=0, \tag{27}$$
$$b-d+f+i=0, \tag{28}$$
$$a-d+g+h=0. \tag{29}$$

Of these, the first five are clearly independent; (21), (22), and (24) give (25); (22), (23), and (25) give (26); (20), (21), and (25) give (27); (22), (23), and (27) give (28); (21), (22), and (28) give (29); or, in other words, there exist the following five independent relations connecting the left-hand members of the above identities, viz.,

$$\left.\begin{aligned}(21)+(22)=(24)+(25)=(28)+(29),\\ (22)+(23)=(25)+(26)=(27)+(28),\\ (20)+(25)=(21)+(27)\end{aligned}\right\}. \tag{30}$$

Hence, there are five, and but five, independent identities of the form (14) connecting the areas of triangles formed by the six points 1, 2, 3, 4, 5, 6 in a plane. Accordingly, Clifford's statement,* "There

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* *Vide loc. cit.*, § 3.

It is interesting to remark here that, by taking the corrected form (14) of the identity (*a*) and permuting cyclically, we obtain the following six identities:

$$\left.\begin{aligned}123.456+124.365+125.634+126.543=0,\\ 234.561+235.416+236.145+231.654=0,\\ 345.612+346.521+341.256+342.165=0,\\ 456.123+451.632+452.361+453.216=0,\\ 561.234+562.143+563.412+564.321=0,\\ 612.345+613.254+614.523+615.432=0\end{aligned}\right\}. \tag{*b*}$$

The sum of the left-hand members of the first, third, and fifth of these identities is

are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123.456+124.563+125.634+126.345\equiv 0, \qquad (a)$$

with five others obtained from this by permutation," requires modification with regard both to the form of the identities and their number.

To return to the original point, the independent absolute invariants of the system of six points in the plane under the general projective group of the plane are thus determined both in form and number.

2. A system of eight points in ordinary space is determined by twenty-four coordinates. The eight points form seventy different tetrahedra. The volumes of these tetrahedra arrange themselves in pairs in thirty-five different products to satisfy twenty-five independent identical relations of the form

$$1234.5678+1235.6784+1236.7845+1237.8456+1238.4567=0. \quad (31)$$

This identity is established in a manner wholly analogous to the method of proving the identity (14) in the plane: the identity is a direct consequence of the determinantal relation

$$\begin{aligned}&|\,a_1, b_2, c_3\,|\,.\,|\,a_4, b_5, c_6\,| - |\,a_2, b_3, c_4\,|\,.\,|\,a_1, b_5, c_6\,|\\ &\quad + |\,a_3, b_4, c_1\,|\,.\,|\,a_2, b_5, c_6\,| - |\,a_4, b_1, c_2\,|\,.\,|\,a_3, b_5, c_6\,| = 0, \qquad (32)\end{aligned}$$

which itself results immediately from the identity already used

$$|\,a_1, b_2,\,|\,.\,|\,a_3, b_4\,| + |\,a_2, b_3\,|\,.\,|\,a_1, b_4\,| + |\,a_3, b_1\,|\,.\,|\,a_2, b_4\,| = 0.$$

The general projective group of ordinary space is a 15-parameter Lie group generated by the following fifteen independent infinitesimal transformations:

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zero, which is also the sum of the left-hand members of the second, fourth, and sixth; hence but four are independent. They contain all possible products except 135.624; if we write down the identity involving this term, and permute cyclically, the system formed of these six identities and of the six (*b*) is found to reduce to eight different identities among which three relations appear, leaving but five independent as before.

$$\left.\begin{array}{c} p \equiv \dfrac{\partial f}{\partial x}, \quad q \equiv \dfrac{\partial f}{\partial y}, \quad r \equiv \dfrac{\partial f}{\partial z}, \\ xp,\ yp,\ zp,\ \ xq,\ yq,\ zq,\ \ xr,\ yr,\ zr, \\ x(xp+yq+zr), \quad y(xp+yq+zr), \quad z(xp+yq+zr). \end{array}\right\} \tag{33}$$

The nine independent ratios of the terms of the identity (31), and of those identities derived from it by permutation, are solutions of the simultaneous system

$$\sum_1^8 i \frac{\partial \phi}{\partial x_i} = \sum_1^8 i \frac{\partial \phi}{\partial y_i} = \sum_1^8 i \frac{\partial \phi}{\partial z_i} = \sum_1^8 i x_i \frac{\partial \phi}{\partial x_i}$$

$$= \ldots = \sum_1^8 i \left\{ z_i \left(x_i \frac{\partial \phi}{\partial x_i} + y_i \frac{\partial \phi}{\partial y_i} + z_i \frac{\partial \phi}{\partial z_i} \right) \right\} = 0, \tag{34}$$

and hence are the invariants of the original system of eight points in space of three dimensions by the projective group of that space.

3. The generalization of the identities (14) and (31), together with a study of the Lie group which represents the general projective group of n-dimensional space, furnishes a basis for the complete generalization of Clifford's statement of the theory of anharmonics for two dimensions.

Consider the expression formed by the sum of the following binary products of determinants:

$$\left.\begin{array}{rl} & |\ 1, 2, 3, 4, 5, \ldots, n, \quad |\ .\ |\ n+1, n+2, n+3, \ldots, 2n\ |\ , \\ (-1)^n & |\ 2, 3, 4, 5, 6, \ldots, n, n+1\ |\ .\ |\ 1, n+2, n+3, \ldots, 2n\ |\ , \\ & |\ 3, 4, 5, 6, \ldots, n, n+1, 1\ |\ .\ |\ 2, n+2, n+3, \ldots, 2n\ |\ , \\ (-1)^n & |\ 4, 5, 6, \ldots, n, n+1, 1, 2\ |\ .\ |\ 3, n+2, n+3, \ldots, 2n\ |\ , \\ & \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots, \\ (-1)^n & |\ n+1, 1, 2, 3, \ldots, n-1\ |\ .\ |\ n, n+2, n+3, \ldots, 2n\ | \end{array}\right\}, \tag{35}$$

where, for brevity,

$$|\ 1, 2, 3, 4, 5, \ldots, n\ | \equiv |\ a_1^{(1)}, a_2^{(2)}, a_3^{(3)}, \ldots, a_n^{(n)}\ |\ . \tag{36}$$

By expanding all the determinants with regard to the elements of their left-hand columns, we see that the coefficient of the general term $a_j^{(r)}$, $a_k^{(s)}$ of the above sum is equal to the sum of determinant products, which sum is of the *form*

$$\begin{aligned}
&| 1, 2, 3, 4, \ldots, n-1 | \, . \, | n, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+(-1)^{n-1} &| 2, 3, 4, \ldots, n-1, n | \, . \, | 1, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+ &| 3, 4, \ldots, n-1, n, 1 | \, . \, | 2, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+(-1)^{n-1} &| 4, \ldots, n-1, n, 1, 2 | \, . \, | 3, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
+(-1)^{n-1} &| n, 1, 2, 3, \ldots, n-2 | \, . \, | n-1, n+1, n+2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad n+3, \ldots, 2(n-1) | \, . \qquad (37)
\end{aligned}$$

By treating this sum (37) in the same manner, its general term is found to have a coefficient of the same form in which n is again decreased by unity; carrying this through $n-2$ steps, we arrive finally at a coefficient of the form

$$| 1, 2 | \, . \, | 3, 4 | + | 2, 3 | \, . \, | 1, 4 | + | 3, 1 | \, . \, | 2, 4 | ,$$

which is obviously zero; hence the original sum of the determinant products (35) is zero, that is,

$$\Sigma | \alpha, \beta, \gamma, \delta, \epsilon, \ldots, \mu | \, . \, | \nu, n+2, n+3, \ldots, 2n | \equiv 0, \qquad (38)$$

where $\alpha\beta\gamma\delta\epsilon \ldots \mu\nu$

takes successively the values of the sequence

$$1, 2, 3, 4, 5, \ldots, n, n+1$$

and its cyclical permutations, and is always positive, or alternately positive and negative, according as n is even or odd.

A system of $[2(n+1)]$ points in a space of n dimensions depends upon $2n(n+1)$ coordinates and forms $[2(n+1)]!/[(n+1)!]^2$ solids

each having $n+1$ vertices. The volumes of these solids satisfy identities of the form

$$\begin{aligned}
&[1, 2, 3, 4, \ldots, n, \quad n+1].[n+2, n+3, n+4, \ldots, 2(n+1)]\\
\pm&[1, 2, 3, 4, \ldots, n, \quad n+2].[n+3, n+4, \ldots, 2(n+1), n+1]\\
+&[1, 2, 3, 4, \ldots, n, \quad n+3].[n+4, \ldots, 2(n+1), n+1, n+2]\\
\pm&[1, 2, 3, 4, \ldots, n, \quad n+4].[n+5, \ldots, 2(n+1), n+1, n+2, n+3]\\
+&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
\pm&[1, 2, 3, 4, \ldots, n, 2(n+1)].[n+1, n+2, n+3, \ldots, 2n+1] \equiv 0, \quad (39)
\end{aligned}$$

the upper or lower signs being taken according as n is odd or even. This identity follows directly from (38). The other identities, $[2(n+1)]!/2[(n+1)!]^2-(n^2+2)$ in number, are derived from this one by permutation.

The general projective group of n-dimensional space is an $[n(n+2)]$ parameter group generated by the following $[n(n+2)]$ independent infinitesimal transformations

$$\frac{\partial f}{\partial x_i}, \quad x_i \frac{\partial f}{\partial x_k}, \quad x_i \sum_1^n j x_j \frac{\partial f}{\partial x_j} \quad (i, k = 1, \ldots, n).$$

The n^2 independent ratios of products of the form of those entering (39) are the $[2n(n+1)-n(n+2)]$ absolute invariants of the system of $[2(n+1)]$ points by the general projective group of n-dimensional space.

Zeroes of the Bessel Functions. *By* H. M. MACDONALD. Communicated (provisionally) April 7th, 1898. Received July 16th, 1898.

The zeroes of any holomorphic function can be approximated to by finding the zeroes of the polynomials obtained by stopping at any term of the power series representing the function. This method indicates the nature of the zeroes, but is unsuitable for the calculation of their values. In the neighbourhood of an essential singularity the function has an infinite number of zeroes, and all these zeroes can be quickly calculated from the expression for the function in that neighbourhood. This is the method applied by Stokes to calculate

the zeroes of $J_0(z)$.* When all the zeroes associated with the essential singularities of the function have been found, there will remain a finite number of zeroes or none. In this paper it is shown, § 1, that the number of zeroes of a holomorphic function $f(z)$ inside the space bounded by a contour for which $|f(z)|$ is constant exceeds the number of the zeroes of the function $f'(z)$ in the same space by unity. When, by successive applications of the theorem, a derived function is found which has no zeroes which are not associated with an essential singularity, the number of zeroes of $f(z)$ not associated with an essential singularity is known. In § 3 the theorem of § 1 is applied to prove the separation of the real zeroes of a function with real coefficients by the real zeroes of its derived function. In § 4 a method is given of calculating the zeroes of $f(z)$, when those of $f'(z)$ are known. It is shown, §§ 5–7, that the zeroes of $J_n(z)/z^n$ are all real and associated with the essential singularity at infinity when n is real and greater than -1, and in §§ 8, 9 it is proved that all the real zeroes of $J_n(z)/z^n$, when n is real and less than -1, and not an integer, are associated with the essential singularity at infinity. When n is a negative integer $-m$, $J_n(z)/z^n$ has, in addition, $2m$ real zeroes coincident at the origin. It is proved in § 10 that, when $n = -m-\nu$, m being a positive integer, and $1>\nu>0$, $J_n(z)/z^n$ has a finite number $2m$ of zeroes which are not associated with the essential singularity at infinity. Formulæ for calculating these $2m$ zeroes are given in § 11. It appears that, n being real, starting with any positive value of n, the zeroes nearest the origin approach it as n diminishes, two of them reaching it when n has diminished to -1, and two more reaching it whenever n passes through a negative integral value; these zeroes then become complex for values of n not integral.

1. The number of zeroes of the function $f(z)$ in the space bounded by a contour at each point of which $|f(z)| = m_1$ exceeds the number of zeroes of the derived function $f'(z)$ in the same space by unity, the function $f(z)$ being holomorphic throughout the space and having no singularity in it.

The number N of zeroes of $f(z)$ inside the contour $|f(z)| = m_1$ is finite, since $f(z)$ has no essential singularity in the space. When m is very small, the portion of the curve $|f(z)| = m$ inside the contour $|f(z)| = m_1$ consists of N closed curves surrounding the

* *Camb. Phil. Trans.*, Vol. IX.

N zeroes of $f(z)$ in the space; as m increases, these ovals increase until two of them coalesce, the point at which they coalesce being a double point on the curve corresponding to that particular value of m. When m has increased to its final value m_1, the N closed curves have coalesced into one closed curve, and therefore $N-1$ double points have been passed through. Each of these double points is a zero of $f'(z)$, for, if $f(z) = \phi + \iota\psi$, where ϕ and ψ are functions of x and y with real coefficients, at a double point on the curve $\phi^2+\psi^2 = \text{const.}$ $\dfrac{\partial\phi}{\partial x}$ and $\dfrac{\partial\psi}{\partial x}$ vanish, that is, $f'(z)$ vanishes. Hence the number of zeroes $f'(z)$ inside the contour is $N-1$. To complete the proof, it should be observed that two ovals cannot coalesce at more than one point, $f(z)$ being single valued. The proof assumes the zeroes of $f(z)$ inside the contour to be all different; the case where $f(z)$ has equal zeroes is immediately reducible to the above by dividing out the factor common to $f(z)$ and $f'(z)$. If $f'(z)$ has two zeroes equal, two of the double points coalesce, that is, three ovals coalesce at the same point.

Similarly, it can be shown that the number of zeroes of $f'(z)$ in the space between the contours $|f(z)| = m_1$ and $|f(z)| = m_2$ is equal to the number of zeroes of $f(z)$ in the same space, $f(z)$ being holomorphic throughout the space and having no singularity* in it.

2. The converses of the above theorems are obviously true, that, if inside a contour for each point of which $|f(z)| = m_1$, $f'(z)$ has $N-1$ zeroes, $f(z)$ has N zeroes; and that, if $f'(z)$ has N zeroes in the space between two contours $|f(z)| = m_1$ and $|f(z)| = m_2$, $f(z)$ has N zeroes in the same space, $f(z)$ being holomorphic throughout the space, and having no singularity in it.

A simple proof of the theorem that a polynomial of the n^{th} degree has n zeroes can be obtained from the above. For let the curve $|f(z)| = m_1$ consist of N' ovals; then, if N is the total number of zeroes of $f'(z)$, there are $N-N'+1$ zeroes of $f'(z)$ inside the N' ovals (there being $N'-1$ double points to be passed through to coalesce the N' ovals). Therefore the total number of zeroes of $f(z)$ is $N-N'+1+N'$, being one more for each oval, that is, $N+1$. Now a polynomial of degree unity has one zero; therefore the polynomial of degree n has n zeroes.

* When $f(z)$ has an essential singularity at infinity, the contours which enclose it are of parabolic form.

3. If a holomorphic function $f(z)$ has all its coefficients real, and all its zeroes real and different, then between two consecutive zeroes of $f(z)$ there is one zero and one only of $f'(z)$.

In this case the curve $|f(z)| = m$ is symmetrical with respect to the real axis, and the double points therefore lie on the real axis, there being one and one only between two consecutive zeroes of $f(z)$, for, if there were more than one, there would be ovals between these two zeroes of $f(z)$ when m is sufficiently diminished; that is, they would not be consecutive zeroes. The theorem is clearly true in the case where all the zeroes of the function inside a contour $|f(z)| = \text{const.}$ are real, the zeroes of $f'(z)$ inside this contour separating them.

4. The zeroes of $f(z)$ can be obtained in the form of a series when those of $f'(z)$ are known.

Let z_0 be a zero of $f'(z)$ and let $f(z_0) = a$; then

$$f(z)-a = \frac{(z-z_0)^2}{2!}f''(z_0)+\frac{(z-z_0)^3}{3!}f'''(z_0)+\&c.;$$

that is, $$f(z)-a = (z-z_0)^2\chi(z).$$

Then the function $f(z)$ has two zeroes continuous with the two zeroes z_0 of $f(z)-a$; they lie inside the contour determined by the condition $|f(z)-a| > |a|$, and are given by

$$z = z_0+\sum_1^\infty \frac{e^{\pm i n\pi i}}{n!}\frac{d^{n-1}}{dz^{n-1}}\left\{\frac{a}{\chi(z)}\right\}^{\frac{1}{n}}.$$

5. The zeroes of the function $J_n(z)/z^n$ are all real when n is real and greater than -1, $J_n(z)$ being Bessel's function. This is usually established by considering $J_n(z)/z^n$ as the limit of a polynomial when its degree is indefinitely increased. It may be proved as follows: $2^n\pi(n)\,J_n(z)/z^n \equiv u_n$ is a holomorphic function both of z and n, when $|n|<1$,* and therefore its zeroes are continuous with those of $u_{\frac{1}{2}}$ or $u_{-\frac{1}{2}}$ (Picard, *Traité d'Analyse*, t. II., chap. ix.), each zero of u_n being given by a Lagrange series, each term of which is real when n is. Therefore all the zeroes of u_n are real when n is real and $1>n>-1$.

Again, $$\frac{d}{dz}\frac{J_n(z)}{z^n} = -\frac{J_{n+1}(z)}{z^n};$$

therefore, by § 3 above, $J_{n+1}(z)/z^n$ has all its zeroes real, there being one and one only between two consecutive zeroes of $J_n(z)/z^n$; that

* u^n is a holomorphic function of n when the real part of n is greater than -1.

is, all the zeroes of $J_{n+1}(z)/z^{n+1}$ are real if the zeroes of $J_n(z)/z^n$ are, and they have been shown to be real when n is real and $1>n>-1$; hence the zeroes of $J_n(z)/z^n$ are all real when n is real and greater than -1.

6. The zeroes of $J_n(z)/z^n$ are separated by the zeroes of $J_{n+1}(z)/z^n$, one zero of $J_{n+1}(z)/z^n$, and one only, lying between two consecutive zeroes of $J_n(z)/z^n$.

This is a particular case of § 3 above; other proofs of it have been given by Hobson, *Proc. Lond. Math. Soc.*, Vol. XXVII.; Van Vleck, *Amer. Jour. of Math.*, Vol. XIX. It can also be established as follows: If α and β are two consecutive real zeroes of $J_n(z)/z^n$, the recurrence formula at once gives

$$\int_\alpha^\beta \frac{J_{n+1}(z)}{z^n}\,dz = 0;$$

and therefore $J_{n+1}(z)/z^n$ vanishes once at least when z is between α and β. Again, if α' and β' are two consecutive real zeroes of $z^{n+1}J_{n+1}(z)$,

$$\int_{\alpha'}^{\beta'} z^{n+1}J_n(z)\,dz = 0;$$

and therefore $z^{n+1}J_n(z)$ vanishes once at least when z is between α' and β'. The two statements taken together show that between any two consecutive real zeroes of $J_n(z)/z^n$ there is one, and only one, zero of $J_{n+1}(z)/z^n$, the zero $z=0$ of $J_{n+1}(z)/z^n$ being the one which separates the least positive and negative zeroes of $J_n(z)/z^n$. The theorem holds for the real zeroes of $J_n(z)/z^n$ whatever real value n has.

7. When n is real and greater than -1, all the zeroes of $J_n(z)/z^n$ are given by Stokes' formula.

The formula for the m^{th} positive zero in order of magnitude is

$$\zeta_m = \alpha - \frac{4n^2-1}{8\alpha} - \frac{4(4n^2-1)(28n^2-31)}{3(8\alpha)^3} - \&c.,$$

where

$$\alpha = \frac{\pi}{4}(2n+4m-1),$$

and n is real and greater than -1.

It follows from the form of this expression that, if all the zeroes of $J_n(z)/z^n$ are given by it, then all those of $J_{n+1}(z)/z^{n+1}$ are given by the corresponding formula when $n+1$ is written for n (§ 6). It is therefore sufficient to show that all the zeroes are given by this

formula when $\frac{1}{2} > n > -1$. All the zeroes are given by it when n is $\frac{1}{2}$ or $-\frac{1}{2}$, they being then the zeroes of $\frac{\sin z}{z}$ and of $\cos z$. Now $J_n(z)/z^n$ is holomorphic in both z and n, when $|n| < 1$; therefore all its zeroes are continuous with those of $\cos z$. Hence all the zeroes of $J_n(z)/z^n$ are given by Stokes' formula when n is real and greater than -1. This may be expressed by saying that all the zeroes of $J_n(z)/z^n$ are associated with the essential singularity at infinity when n is real and greater than -1.

8. The least positive real zero of $z^n J_{-n}(z)$ is less than the least positive real zero of $J_n(z)/z^n$, and less than the least positive real zero of $z^{n+1}J_{-n-1}(z)$, n being real and positive.

The first result follows at once from the relation between $J_n(z)$ and $J_{-n}(z)$, which are independent solutions of Bessel's differential equation, there being a zero of $z^n J_{-n}(z)$ between the origin and the least positive zero of $J_n(z)/z^n$. Writing

$$u_n = \left(\frac{2}{z}\right)^n J_n(z),$$

the relation between u_n and u_{-n} is

$$\frac{z^2}{4} u_{n+1} u_{-n} + u_n u_{-n-1} = -\frac{\sin n\pi}{\pi}.$$

Now, when $z = 0$,

$$u_n = \frac{1}{\Pi(n)} \quad \text{and} \quad u_{-n-1} = -\frac{\sin n\pi}{\pi}\Pi(n),$$

and, when z is equal to z_0, the least positive zero of u_{-n}, u_n is positive, and therefore u_{-n-1} has the same sign as it has when $z = 0$; hence the least positive real zero of u_{-n} is less than the least positive real zero of u_{-n-1}. Similar results hold for the negative real zero of u_{-n} nearest to the origin.

9. All the real zeroes of u_{-n}, when n is real and positive, are associated with the essential singularity at infinity.

The formula for the zeroes, which corresponds to Stokes' formula for n positive, and which is derived from the semi-convergent form of $J_{-n}(z)$, is easily shown to be the same as that quoted, § 7, where a is now $\frac{\pi}{4}(-2n+4k+4m-1)$, k being the greatest positive integer less than n. It is obvious that the zeroes of u_{-n} given by this formula separate those given by the corresponding formula for the

zeroes of u_n, the least positive zero of u_{-n} being less than the least positive zero of u_n, whence it follows (as there can only be one zero of u_{-n} between two consecutive real zeroes of u_n) that all the real zeroes of u_{-n} are given by the formula; that is, they are all associated with the essential singularity at infinity.

10. When $m+1>n>m$, m being an integer, $z^n J_{-n}(z)$ has $2m$ complex zeroes, which, when m is even, are arranged in $m/2$ groups of four, each zero of a group having the same modulus, and, when m is odd, in $(m-1)/2$ such groups of four together with two pure imaginary zeroes.

Writing, as before, $$u_n = \left(\frac{2}{z}\right)^n J_n(z),$$

the relation between u_n and u_{-n} is

$$\frac{z^2}{4} u_{n+1} u_{-n} + u_n u_{-n-1} = -\frac{\sin n\pi}{\pi}.$$

When $1>n>0$, putting $z = \iota y$ (y being real), u_{-n} is positive for all values of y, and increases with y. Now

$$\frac{\partial u_{-n-1}}{\partial y} = 2y u_{-n};$$

therefore u_{-n-1} increases with y. Again u_{-n-1} is negative when $y=0$, and positive when $y=\infty$; therefore u_{-n-1} changes sign once as z goes from 0 to $\iota\infty$; that is, u_{-n-1} has one pure imaginary zero between $z=0$ and $z=\iota\infty$. Hence, when $2>n>1$, u_{-n} has a pair of pure imaginary zeroes which are equal and of opposite sign. When $2>n>1$, u_{-n-1} is positive when $z=0$, and decreasing for u_{-n} is then negative; therefore u_{-n-1} decreases to a minimum when u_{-n} vanishes, and then increases indefinitely with y. Now, when u_{-n} vanishes,

$$u_{-n-1} = -\frac{\sin n\pi}{\pi u_n},$$

a positive quantity, for u_n is positive and $\sin n\pi$ is negative; therefore u_{-n-1} is positive for all pure imaginary values of z. Hence, when $3>n>2$, u_{-n} has no pure imaginary zeroes. Similarly it follows that, when $4>n>3$, u_{-n} has a pair of pure imaginary zeroes, and, when $5>n>4$, u_{-n} has no pure imaginary zeroes. Therefore, when $2m+1>n>2m$, u_{-n} has no pure imaginary zeroes, and, when $2m+2>n>2m+1$, u_{-n} has a pair of equal imaginary zeroes of opposite sign.

Since the least positive real zero of u_{-n+1} is less than that of u_{-n},

$$| u_{-n} |_{z=x_0} > | u_{-n} |_{z=0},$$

where x_0 is the least positive real zero of u_{-n+1}. When u_{-n} has no pure imaginary zero, u_{-n} is positive when $z=0$, and increases as z goes from 0 to $\iota\infty$; therefore u_{-n} takes the value $| u_{-n} |_{z=x_0}$ at some point on the positive imaginary axis, and also at an equidistant point on the negative imaginary axis. When u_{-n} has pure imaginary zeroes, it is negative at the origin, and its modulus decreases as z goes from 0 to an imaginary zero of u_{-n} along the axis of imaginaries, and then increases indefinitely; therefore u_{-n} will take the value $| u_{-n} |_{z=x_0}$ at two points on the axis of imaginaries equidistant from the origin, and further from it than the pure imaginary zeroes. Let the contour be drawn for each point of which

$$| u_{-n} | = | u_{-n} |_{z=\pm x_0};$$

the portion of it between $z=-x_0$ and $z=x_0$ will, by the above, surround the origin; hence there is a closed curve, enclosing the origin, which is part of a contour for which $| u_{-n} |$ is slightly less than $| u_{-n} |_{z=\pm x_0}$. The number of zeroes of u_{-n} inside this closed curve will, by § 1, exceed the number of zeroes of $\frac{du_{-n}}{dz}$ inside it by unity. Now

$$\frac{du_{-n}}{dz} = -2zu_{-n+1},$$

and $z=0$ is a zero of zu_{-n+1}; therefore the number of zeroes inside the closed curve exceeds the number of zeroes of u_{-n+1} inside it by 2. It has been shown that the two real zeroes of u_{-n+1} nearest the origin are nearer the origin than the two nearest real zeroes of u_{-n}; therefore the zeroes of u_{-n} and u_{-n+1} inside this contour are complex. Now, when $1>n>0$, u_{-n} has no zeroes which are not real, therefore, when $2>n>1$, u_{-n} has two zeroes which are not real; hence, when $m+1>n>m$, m a positive integer, u_{-n} has $2m$ zeroes which are not real. Since u_{-n} involves only even powers of z, and all its coefficients are real, if it has a pure imaginary zero, it must have an equal pure imaginary zero with opposite sign; and, if it has a complex zero, it must have an equal complex zero with opposite sign, and two other zeroes conjugate to these two, forming a group of four with the same modulus. Hence the theorem is completely established. The number of the complex zeroes of u_{-n} has been obtained by Hurwitz, *Math. Ann.*, Vol. XXXIII., p. 246, by a different method.

11. *Investigation of a Formula for the Complex Zeroes of* $J_n(z)/z^n$. It has been shown above that, when $m = n+\mu$, where m is a positive integer, and $1>\mu>0$, u_{-n} has $2m$ complex zeroes inside a certain closed curve. When $\mu = 0$, these $2m$ zeroes become the $2m$ zeroes of $x^{2m}u_m$ inside the closed curve, and these are the $2m$ zeroes of $x^{2m} = 0$; therefore, when $n = m+\mu$, the $2m$ complex zeroes of u_{-n} are continuous with the zeroes of $x^{2m} = 0$. To obtain an expression for these zeroes it is necessary to express u_{-n} in a suitable form. The expression for u_{-n} in a series is given by

$$u_{-n} = \sum_{s=0}^{s=\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(-n+s)};$$

that is,
$$u_{-n} = \sum_{s=0}^{s=\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(-m-\mu+s)};$$

whence
$$u_{-n} = \sum_{0}^{m-1} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(-m-\mu+s)} + \sum_{m}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(s-m-\mu)}.$$

Now, $\Pi(-m-\mu+s)\,\Pi(m+\mu-s-1) = \pi \operatorname{cosec}(m+\mu-s)\,\pi$;

therefore

$$u_{-n} = (-)^m \frac{\sin \mu\pi}{\pi} \sum_{0}^{m-1} \left(\frac{x^2}{4}\right)^s \frac{\Pi(m+\mu-s-1)}{\Pi(s)} + \sum_{0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^{m+s}}{\Pi(s-\mu)\,\Pi(m+s)}.$$

Hence the zeroes of u_{-n} are those of the expression

$$\frac{\sin \mu\pi}{\pi} X + x^{2m} F,$$

where
$$X = \sum_{0}^{m-1} \left(\frac{x^2}{4}\right)^s \frac{\Pi(m+\mu-s-1)}{\Pi(s)}$$

and
$$F = \frac{1}{2^{2m}} \sum_{0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s-\quad)\,\Pi(m+s)},$$

the zeroes associated with the essential singularity at infinity being continuous with those of $F = 0$, and the complex zeroes (those associated with the essential singularity at the origin) being given by

$$x = \gamma \left\{ \frac{\sin \mu\pi}{\pi} \frac{X}{F} \right\}^{\frac{1}{2m}},$$

where $$\gamma = e^{\pm[(2r+1)/2m]\pi i},$$

r having values 0 to $m-1$ inclusive.

Expanding by Lagrange's theorem,

$$x = \sum_1^\infty \frac{\gamma^k}{k!}\left(\frac{\sin\mu\pi}{\pi}\right)^{k/2m}\left\{\left(\frac{d}{dx}\right)^{k-1}\left(\frac{X}{F}\right)^{k/2m}\right\}_{x=0},$$

or, since X and F only involve even powers of x,

$$x = \sum_0^\infty \frac{\gamma^{2k+1}}{2k+1!}\left(\frac{\sin\mu\pi}{\pi}\right)^{(2k+1)/2m}\left\{\left(\frac{d}{dx}\right)^{2k}\left(\frac{X}{F}\right)^{(2k+1)/2m}\right\}_{x=0}.$$

Writing $$X = F\sum_0^\infty \frac{a_s}{s!}\left(\frac{x^2}{4}\right)^s,$$

$$x = \sum_0^\infty \frac{\gamma^{2k+1}}{2^{2k}(2k+1)\,k!}\left(\frac{\sin\mu\pi}{\pi}\right)^{(2k+1)/2m}\Delta^k a_0^{(2k+1)/2m},$$

where $$\Delta a_s^p = p a_s^{p-1} a_{s+1}.$$

The above series for the roots converges quickly when μ is small, but the convergence becomes very slow when μ is nearly unity. For values of μ near to unity it is better to use another series obtained as follows: The expression u_{-n} can be written in the form

$$X_1 - x^{2m+2}F_1,$$

where $$F_1 = \frac{1}{2^{2m+2}}\sum_0^\infty\left(-\frac{x^2}{4}\right)^s \frac{1}{\Pi(s-\mu+1)\,\Pi(s+m+1)},$$

and X_1 is a polynomial of degree $2m$ in x. The complex zeroes of u_{-n} are now given by

$$x = \gamma\left(\frac{X_1}{F_1}\right)^{1/(2m+2)},$$

where $$\gamma = \pm e^{r\pi i/(m+1)},$$

and r has values 1 to m inclusive; that is,

$$x = \sum_0^\infty \frac{\gamma^{2k+1}}{2^{2k}(2k+1)\,k!}\Delta^k b_0^{(2k+1)/(2m+2)},$$

where $$X_1 = F_1\sum_0^\infty \frac{b_s}{s!}\left(\frac{x^2}{4}\right)^s.$$

This series is in ascending powers of $(1-\mu)^{1/(2m+2)}$, and is rapidly convergent when μ is nearly unity. The zeroes which correspond to $\gamma = \pm 1$ are the two real zeroes of u_{-n} nearest the origin.

The Catenary, and Associated Trajectory, on the Paraboloid, the Cone, &c. *By* A. G. GREENHILL. Read March 12th, 1896. Received August 10th, 1896. Received, in revised form, February 13th, 1898.

In a preceding article (*Proc. Lond. Math. Soc.*, Vol. XXVII) the properties of the catenary curve assumed by a chain wrapped on a sphere have been considered, concluding with an investigation of the shape of the curve when the sphere is spinning about a vertical axis with such rapidity that the influence of gravity may be left out of account.

But, as the analytical results of this last problem are practically the same as those required for the catenary curve of a chain wrapped on a vertical paraboloid of revolution, we resume the investigations and extend them to the allied problems of the catenary on the cone, &c., and, at the same time, consider the associated problems of the motion of a particle on a surface of revolution.

The theory is illustrated at length by working out the simplest *pseudo-elliptic* cases, by means of which the construction of a catenary or trajectory is made to depend upon tabular matter in mathematical tables, in conjunction with the tables for $F\phi$ and $E\phi$, given in Legendre's *Fonctions Elliptiques*, t. II.

The analysis required in these applications has been developed in papers in the *Proceedings of the London Mathematical Society*:—

"Pseudo-Elliptic Integrals and their Dynamical Applications," Vol. XXV;

"The Dynamics of a Top," Vols. XXVI and XXVII;

"The Spherical Catenary," Vol. XXVII;

"The Transformation and Division of Elliptic Functions," Vol. XXVII;

and, to save repetition, the results are quoted in the sequel, with a reference to the volume and the page, as (*L.M.S.*, XXV, p. 195), &c.

The following Dissertations discuss the same subject:—

Bertram, *Diss.*, Marburg, 1876.
Osswald, J., *Diss.*, Freiburg, 1876.
Neumann, L., *Diss.*, Freiburg, 1878.
Schönlicht, L., *Diss.*, Freiburg, 1884.
Sonntag, *Diss.*, Marburg, 1886.

It will be noticed that in all these dynamical applications which require the elliptic integral of the third kind the Jacobian parameter is a fraction of the *imaginary* period, so that the integrals are of the *circular* form, in Legendre's terminology; and tables of the Jacobian Θ functions, even if accessible to us, would be practically useless for our purposes.

But, by choosing as parameters the rational fractions of the imaginary period, beginning with the simplest fractions, we are able to utilize the pseudo-elliptic cases worked out in *L.M.S.*, xxv, to explore the analytical field with a number of well determined particular cases.

A dynamical desideratum appears, then, to be the tabulation of the function

$$\left\{\frac{\Theta\left(u-\frac{rK'i}{n}\right)}{\Theta\left(u+\frac{rK'i}{n}\right)}\right\}^n,$$

where $n = 2, 3, 4, 5, 6, \ldots,$

in the form $$\frac{A+iB}{A-iB},$$

where A and B are single-valued elliptic functions of u, these being the functions analogous to $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, mentioned by Halphen, *Fonctions Elliptiques*, I, p. 222.

It is easy to translate the pseudo-elliptic results of *L.M.S.*, xxv, into this new notation: thus, for instance, from p. 212,

$$\left\{\frac{\Theta(u-\tfrac{1}{2}K'i)}{\Theta(u+\tfrac{1}{2}K'i)}\right\}^2 = \frac{\operatorname{cn} u \operatorname{dn} u + i(1+u)\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u - i(1+u)\operatorname{sn} u};$$

and, from p. 218,

$$e^{3iI(\frac{1}{3}\omega_3)} = \left\{\frac{\Theta(u-\frac{2}{3}K'i)}{\Theta(u+\frac{2}{3}K'i)}\right\}^3$$

$$= \frac{(1-c+c^2)s-(c-c^2)^2+\frac{1}{2}i\sqrt{S}}{(1-c+c^2)s-(c-c^2)^2-\frac{1}{2}i\sqrt{S}},$$

$$e^{3iI(\frac{1}{3}\omega_2)} = \left\{\frac{\Theta(u-\frac{1}{3}K'i)}{\Theta(u+\frac{1}{3}K'i)}\right\}^3$$

$$= \frac{(1+c)(2-c)\sqrt{(s_1-s.s_2-s)}+i(s-2+c-c^2)\sqrt{(s-s_3)}}{(1+c)(2-c)\sqrt{(s_1-s.s_2-s)}-i(s-2+c-c^2)\sqrt{(s-s_3)}},$$

and so on.

1. Taking Oz as the vertical axis of revolution, and r, ψ the polar coordinates of the projection on a horizontal plane of a point P at a height z, then the statical equations of equilibrium of the chain are

$$T = w(z-h) \text{ or } w(h-z)$$
$$= w(z \sim h), \tag{1}$$

and
$$Tr^2 \frac{d\psi}{ds'} = H, \tag{2}$$

connecting s' the arc, T the tension, and w the weight per unit length of the chain.

Eliminating T, we obtain a general equation, of the form

$$(z \sim h)\, r^2 \frac{d\psi}{ds'} = A, \tag{3}$$

where
$$H = wA. \tag{4}$$

Thence
$$\frac{ds'^2}{dz^2} = 1 + \frac{dr^2}{dz^2} + r^2 \frac{d\psi^2}{dz^2} = \frac{(z-h)^2}{A^2} r^4 \frac{d\psi^2}{dz^2},$$

or
$$\frac{d\psi^2}{dz^2} = A^2 \frac{1 + \frac{dr^2}{dz^2}}{r^2 Z}, \tag{5}$$

where
$$Z = r^2 (z-h)^2 - A^2. \tag{6}$$

The Catenary on the Paraboloid.

2. In the paraboloid of revolution, we put

$$r^2 = 4az, \tag{7}$$

so that
$$\frac{d\psi^2}{dz^2} = \frac{A^2}{4a} \frac{z+a}{z^2 Z}, \tag{8}$$

where
$$Z = 4az(z-h)^2 - A^2; \tag{9}$$

so that, putting
$$A^2 = 4a^2 k^2, \tag{10}$$

$$\frac{d\psi}{dz} = \tfrac{1}{2} k \frac{\sqrt{(z+a)}}{z\sqrt{\{z(z-h)^2 - ak^2\}}}. \tag{11}$$

Then, if p denotes the perpendicular from the origin upon the tangent in the projection of the catenary on a horizontal plane,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \frac{dr^2}{d\psi^2}$$
$$= \frac{(z-h)^2 + k^2}{4ak^2(z+a)}, \tag{12}$$

which can be written

$$4a\frac{k^2}{p^2} = z+a-2(a+h)+\frac{(a+h)^2+k^2}{z+a}$$
$$= \left[\sqrt{(z+a)}-\frac{\sqrt{\{(a+h)^2+k^2\}}}{\sqrt{(z+a)}}\right]^2+2\sqrt{\{(a+h)^2+k^2\}}-2(a+h) \tag{13}$$

showing that p is stationary, and there is a point of inflexion, where

$$z+a = \sqrt{\{(a+h)^2+k^2\}}, \tag{14}$$

and then
$$p^2 = 2a\left[\sqrt{\{(a+h)^2+k^2\}}+(a+h)\right], \tag{15}$$
$$r^2 = 4a\sqrt{\{(a+h)^2+k^2\}}-4a^2. \tag{16}$$

If the paraboloid is spinning about its axis with uniform angular velocity n, then equation (1) must be replaced by

$$T = w(z\sim h)-\frac{wn^2r^2}{g}+C, \tag{17}$$

but, as this equation can be written in the form

$$T = w'(z\sim h'), \tag{18}$$

where
$$w' = w\pm\frac{4wn^2a}{g}, \tag{19}$$

the equations of equilibrium remain essentially the same as before.

It is also immaterial whether we suppose the vertex of the paraboloid to be its highest or lowest point, as an alteration of the direction of gravity with respect to the surface merely changes the sign of T.

With r^2 for independent variable, equation (11) must be written

$$\psi = \tfrac{1}{2}\int\frac{8a^2k\sqrt{(r^2+4a^2)}}{r^2\sqrt{\{r^2(r^2-4ah)^2-64a^4k^2\}}}dr^2. \tag{20}$$

The arc s' is given by the equation

$$\frac{ds'}{dz} = \frac{ds}{r^2d\psi}\frac{r^2d\psi}{dz} = \frac{z-h}{A}\tfrac{1}{2}k\frac{r^2}{z}\frac{\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}}$$
$$= \frac{(z-h)\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}}, \tag{21}$$

or
$$s' = \int\frac{(z-h)\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}} \tag{22}$$
$$= \int\frac{(r^2-4ah)\sqrt{(r^2+4a^2)}}{\sqrt{\{r^2(r^2-4ah)^2-64a^4k^2\}}}dr^2. \tag{23}$$

3. Equations (11) and (15) are elliptic integrals of the third kind, with a single pole at the vertex of the paraboloid, so that they can be compared immediately with the standard form

$$I(v) = \tfrac{1}{2}\int \frac{P(s-\sigma)+\sqrt{(-\Sigma)}}{(s-\sigma)\sqrt{S}}\,ds, \tag{A}$$

where
$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2, \tag{B}$$

$$M^2(s-\sigma) = \wp u - \wp v, \tag{C}$$

$$\int \frac{ds}{\sqrt{S}} = Mu, \tag{D}$$

$$e^{iI(v)} = \frac{\sigma(u+v)}{\sigma(u-v)} e^{(MP-\zeta\sigma)u}. \tag{E}$$

Writing equation (11)

$$\psi = \tfrac{1}{2}k\int \frac{z+a}{z\sqrt{Z}}dz, \tag{24}$$

where
$$Z = (z+a)\{z(z-h)^2 - ak^2\}$$
$$= z^4 + (a-2h)z^3 - h(2a-h)z^2 + a(h^2-k^2)z - a^2k^2, \tag{25}$$

the comparison with (A) is made by putting

$$\frac{Nz}{z+a} = \wp u - \wp v = M^2(s-\sigma), \tag{26}$$

making $u = 0, v, c$, correspond to $z = -a, 0, \infty$; and then we find

$$\wp u = \frac{\frac{1}{6}(a+h)(3a+h)z - \frac{1}{3}ah(a+h) - \frac{1}{2}ak^2}{2(z+a)}, \tag{27}$$

$$\wp' u = -a\frac{\frac{1}{4}(a+h)^2 + \frac{1}{4}k^2}{(z+a)^2}\sqrt{Z}. \tag{28}$$

Denoting the roots of the cubic factor of Z,

$$z(z-h)^2 - ak^2 = 0, \tag{29}$$

by z_1, z_2, z_3, these roots are essentially positive; and they may be written in the order

$$\infty > z > z_1 > z_2 > z > z_3 > 0 > -a > -\infty, \tag{30}$$

so that we may put

$$v = f\omega_3, \quad c = f'\omega_1, \tag{31}$$

and the determination of the arc s', depending on the parameter c, will be found to lead to an analysis of the same nature as that already developed for the Spherical Catenary.

Equation (24) may be written

$$\psi = \tfrac{1}{2}\int \frac{Nk}{\wp u - \wp v}\,\frac{dz}{\sqrt{Z}}$$

$$= \tfrac{1}{2}\int \frac{i\wp' v}{\wp u - \wp v}\,du,$$

or
$$\psi i = -u\zeta v + \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}. \qquad (32)$$

Expressed as a function of s and $\sqrt{S}$, as in the standard form (A),

$$\psi = \tfrac{1}{2}\int \frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}P\int \frac{ds}{\sqrt{S}} - I(v)$$

$$= \tfrac{1}{2}PMu - I(v), \qquad (33)$$

and
$$u = \int \frac{dz}{\sqrt{Z}} = \frac{2F\phi}{\sqrt{(z_1 - z_3 . z_2 + a)}}, \qquad (34)$$

where

	In the open branch.	In the limited branch.	
$\sin^2\phi =$	$\dfrac{z_2+a}{z_1+a}\,\dfrac{z-z_1}{z-z_2}$,	$\dfrac{z_3+a}{z_2-z_3}\,\dfrac{z-z_3}{z+a}$,	(35)
$\cos^2\phi =$	$\dfrac{z_1-z_2}{z_1+a}\,\dfrac{z+a}{z-z_2}$,	$\dfrac{z_3+a}{z_2-z_3}\,\dfrac{z_2-z}{z+a}$,	(36)
$\Delta^2\phi =$	$\dfrac{z_1-z_2}{z_1-z_3}\,\dfrac{z-z_3}{z-z_2}$,	$\dfrac{z_3+a}{z_1-z_3}\,\dfrac{z_1-z}{z+a}$,	(37)

taking z, ψ, and ϕ as increasing together, to avoid ambiguities of sign.

Thus, in the pseudo-elliptic cases, the catenary on the paraboloid cannot be an algebraical curve unless we can make $P(v)$ vanish.

When $P(v) = 0$, the discriminant of S is negative, and the catenary has one open branch only; and now, with z_1 and z_2 imaginary,

$$u = \int \frac{dz}{\sqrt{Z}} = \frac{F\phi}{\sqrt{(HH')}}, \qquad (38)$$

where
$$H^2 = z_3 - z_1 . z_3 - z_2, \quad H'^2 = a + z_1 . a + z_2, \qquad (39)$$

and
$$\tan^2\tfrac{1}{2}\phi = \frac{H}{H'}\,\frac{z-z_3}{z+a}. \qquad (40)$$

4. To express the curve made by the projection of the chain on a horizontal plane, we must return to the standard integral (A) of the circular form; and now, putting

$$s-\sigma = \frac{Qz}{z+a} = \frac{Qr^2}{r^2+4a^2} = Q\sin^2\omega, \tag{41}$$

where ω is the angle the normal makes with the axis, then

$$\frac{r^2}{4a^2} = \frac{z}{a} = \frac{s-\sigma}{Q-s+\sigma}. \tag{42}$$

Denoting the roots of (B) by s_1, s_2, s_3, and replacing

$$s_1-\sigma,\ s_2-\sigma,\ s_3-\sigma \text{ by } \sigma_1,\ \sigma_2,\ \sigma_3,$$

the roots of
$$s^3+3s^2\varphi v+\tfrac{1}{2}s\varphi''v+\tfrac{1}{4}\varphi'^2v = 0, \tag{43}$$

then
$$\frac{z_1}{a} = \frac{\sigma_1}{Q-\sigma_1},\ \frac{z_2}{a} = \frac{\sigma_2}{Q-\sigma_2},\ \frac{z_3}{a} = \frac{\sigma_3}{Q-\sigma_3}; \tag{44}$$

and, since z_1, z_2, z_3 are the roots of the cubic equation

$$z(z-h)^2-ak^2 = 0, \tag{45}$$

therefore
$$\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3} = 0, \tag{46}$$

or
$$\sqrt{\left(\frac{\sigma_1}{Q-\sigma_1}\right)}+\sqrt{\left(\frac{\sigma_2}{Q-\sigma_2}\right)}+\sqrt{\left(\frac{\sigma_3}{Q-\sigma_3}\right)} = 0, \tag{47}$$

an equation for determining Q; rationalizing this equation,

$$\sigma_1^2(Q-\sigma_2)^2(Q-\sigma_3)^2+\ldots-2\sigma_2\sigma_3(Q-\sigma_1)^2(Q-\sigma_2)(Q-\sigma_3)-\ldots = 0, \tag{48}$$

and putting
$$\sigma_1+\sigma_2+\sigma_3 = S_1 = -3\varphi v, \tag{49}$$

$$\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2 = S_2 = \tfrac{1}{2}\varphi''v, \tag{50}$$

$$\sigma_1\sigma_2\sigma_3 = S_3 = -\tfrac{1}{4}\varphi'^2v, \tag{51}$$

$$(S_1^2-4S_2)Q^4+12S_3Q^3-6S_1S_3Q^2+4S_2S_3Q-3S_3^2 = 0, \tag{52}$$

a quartic equation in which the quadrinvariant vanishes; which can therefore be resolved, and it has two real and two imaginary roots.

Putting
$$Q = \frac{3S_3}{R+S_2}, \tag{53}$$

$$R^4-6aR^2-8bR-3a^2 = 0, \tag{54}$$

where
$$a = S_2^2-3S_1S_3 = \tfrac{1}{4}\varphi''^2-\tfrac{9}{4}\varphi\varphi'^2, \tag{55}$$

$$2b = 2S_2^3+27S_3^2-9S_1S_2S_3$$
$$= \tfrac{1}{4}\varphi'^3+\tfrac{27}{16}\varphi'^4-\tfrac{27}{8}\varphi\varphi'^2\varphi'', \tag{56}$$

and (54) is a Jacobian quartic the roots of which, on putting

$$b^2-a^3=c^3, \tag{57}$$

can be exhibited in the form

$$R=\surd(a+c)+\surd\{2a-c+2\surd(a^2-ac+c^2)\},$$

or

$$R=\surd(a+c)+\surd(a+\omega c)+\surd(a+\omega^2 c), \tag{58}$$

where ω, ω^2 denote the imaginary cube roots of unity.

In the sequel it will be sufficient, in general, to work with the parameter

$$v=\frac{2\omega_3}{\mu}, \tag{59}$$

and thus take

$$\sigma=-x, \quad S_1=\tfrac{1}{4}(y+1)^2+x, \quad S_2=\tfrac{1}{2}x(y+1), \quad S_3=\tfrac{1}{4}x^3, \tag{60}$$

when

$$a=\tfrac{1}{16}x^2\{(y+1)^2-12x\}, \tag{61}$$

$$b=-\tfrac{1}{64}x^3\{(y+1)^3+36x(y+1)-54x\}, \tag{62}$$

$$c^3=\frac{27x^4\Delta}{1024}, \tag{63}$$

where Δ denotes the discriminant of the cubic S in (B).

5. In a transformation of even order we may assume that a root of the cubic $S=0$, say $s=s_3$, and at the same time that a root of the cubic (29) is known, say z_3 or r_3, in the form

$$\frac{z_3}{a}=\frac{r_3^2}{4a^2}=4\gamma^2, \tag{64}$$

and then the other two roots will be given in the form

$$\frac{z_1,\ z_2}{a}=\frac{r_1^2,\ r_2^2}{4a^2}=(\beta\pm\gamma)^2, \tag{65}$$

and, with $\beta>3\gamma$, we have

$$r_1=r_2+r_3, \tag{66}$$

and the roots arranged in the order

$$r_1>r_2>r_3.$$

Now, from the general relation (41), or

$$\frac{a}{z}=\frac{4a^2}{r^2}=\cot^2\omega=\frac{Q}{s-\sigma}-1, \tag{67}$$

$$\frac{1}{(\beta+\gamma)^2} = \frac{Q}{s_1-\sigma} - 1, \tag{68}$$

$$\frac{1}{(\beta-\gamma)^2} = \frac{Q}{s_2-\sigma} - 1, \tag{69}$$

$$\frac{1}{4\gamma^2} = \frac{Q}{s_3-\sigma} - 1, \tag{70}$$

and then

$$\frac{(\beta-\gamma)(\beta+3\gamma)}{4\gamma^2(\beta+\gamma)^2} = Q\frac{s_1-s_3}{(s_1-\sigma)(s_3-\sigma)}, \tag{71}$$

$$\frac{(\beta+\gamma)(\beta-3\gamma)}{4\gamma^2(\beta-\gamma)^2} = Q\frac{s_2-s_3}{(s_2-\sigma)(s_3-\sigma)}, \tag{72}$$

$$\left(\frac{\beta-\gamma}{\beta+\gamma}\right)^2\frac{\beta+3\gamma}{\beta-3\gamma} = \frac{s_1-s_3}{s_2-s_3}\frac{s_2-\sigma}{s_1-\sigma}$$

$$= \frac{s_1-\sigma'}{s_2-\sigma'} = \mathrm{dn}^2(1-f)K', \tag{73}$$

where σ' is the value of s corresponding to

$$u = (1-f)\omega_3; \tag{74}$$

thence Q can be determined when $s_1, s_2, s_3, \sigma, \sigma'$ are given, and when the ratio β/γ has been found, by means of the Jacobian quartic equation, implied in (73).

6. In the degenerate case of the catenary on a paraboloid, when

$$\beta = 3\gamma, \tag{75}$$

then, putting

$$4a\gamma^2 = a, \quad h = 3a, \quad ak^2 = 4a^2, \tag{76}$$

$$Z = (z+a)(z-a)^2(z-4a), \tag{77}$$

$$\frac{d\psi}{dz} = \tfrac{1}{2}a\sqrt{\frac{a}{a}}\,\frac{z+a}{z(z-a)\sqrt{\{(z+a)(z-4a)\}}}$$

$$= \tfrac{1}{2}\left(\frac{-\sqrt{(aa)}}{z} + \sqrt{\frac{a}{a}}\,\frac{a+a}{z-a}\right)\frac{1}{\sqrt{\{(z+a)(z-4a)\}}}, \tag{78}$$

$$\psi = \tfrac{1}{2}\sin^{-1}2\sqrt{\left(\frac{a(z+a)}{(a+4a)z}\right)} - \sqrt{\left(\frac{a+a}{3a}\right)}\sin^{-1}\sqrt{\left(\frac{3a}{a+4a}\,\frac{z+a}{z-4a}\right)}, \tag{79}$$

or, with

$$r^2 = 4az, \quad b^2 = 4aa, \tag{80}$$

$$\psi = \tfrac{1}{2}\sin^{-1}\frac{b}{r}\sqrt{\left(\frac{r^2+4a^2}{a^2+b^2}\right)} - \tfrac{1}{2}\sqrt{\left(\frac{b^2+4a^2}{3a^2}\right)}\sin^{-1}\tfrac{1}{2}\sqrt{\left(\frac{3b^2}{a^2+b^2}\,\frac{r^2+4a^2}{r^2-b^2}\right)}. \tag{81}$$

Also
$$\frac{ds'}{dz} = \frac{(z-3a)\sqrt{(z+a)}}{(z-a)\sqrt{(z-4a)}}, \tag{82}$$

so that the catenary is rectifiable in the form

$$s' = \sqrt{(z+a.z-4a)} + a\,\mathrm{th}^{-1}\sqrt{\left(\frac{z-4a}{z+a}\right)} - 4\sqrt{\left(\frac{a.a+a}{3}\right)}\tan^{-1}\sqrt{\left(\frac{3a}{a+a}\,\frac{z+a}{z-4a}\right)}. \tag{83}$$

The Whirling Spherical Catenary.

7. In the catenary curve on a sphere due to centrifugal whirling (*L.M.S.* XXVII, p. 181),
$$\psi = \tfrac{1}{2}\int \frac{A\,dr^2}{r^2\sqrt{R}}, \tag{1}$$

where
$$R = (1-r^2)\left\{r^2(r^2-b)^2 - A^2\right\}$$
$$= (1-r^2)(r^2-r_1^2)(r^2-r_2^2)(r^2-r_3^2), \tag{2}$$

suppose; in which we can take
$$r_1 = r_2 + r_3, \tag{3}$$

in consequence of which relation we encounter the same Jacobian quartic (54) as with the catenary on the paraboloid.

We reduce (1) to the standard form (A), by putting
$$s-\sigma = s+x = \frac{Q'r^2}{1-r^2} = Q'\tan^2\theta, \tag{4}$$

where θ denotes the co-latitude of a point; and now
$$S = 4\,\frac{(Q'+x)\,r^2-x}{1-r^2}\,\frac{Q'^2r^4}{(1-r^2)^2} - \left\{(1+y)\,\frac{Q'r^2}{1-r^2} - x\right\}^2$$

has to assume the form
$$S = \frac{r^2(r^2-b)^2-A^2}{N^2(1-r^2)^3}, \tag{5}$$

and this leads to the Jacobian quartic for the determination of Q', namely,
$$\left[\left\{(1+y)^2+4x\right\}Q'^2+4x(1+y)Q'+3x^2\right]^2 - 4\left\{2x(1+y)Q'+3x^2\right\}\left[4Q'^3+\left\{(1+y)^2+4x\right\}Q'^2+2x(1+y)Q'+x^2\right] = 0. \tag{6}$$

Next,
$$ds = \frac{Q'dr^2}{(1-r^2)^2}, \tag{7}$$

$$\frac{ds}{\sqrt{S}} = \frac{NQ'dr^2}{\sqrt{R}}, \tag{8}$$

and, putting $r = 0$,

$$x = \sqrt{(-\Sigma)} = \frac{A}{N}, \tag{9}$$

$$\frac{\sqrt{(-\Sigma)}}{s-\sigma} = \frac{A}{NQ'}\,\frac{1-r^2}{r^2}, \tag{10}$$

so that
$$I = \tfrac{1}{2}PNQ'\int\frac{dr^2}{\sqrt{R}} - \tfrac{1}{2}A\int\frac{1-r^2}{r^2}\,\frac{dr^2}{\sqrt{R}}$$

$$= \tfrac{1}{2}N(PQ'+x)\int\frac{dr^2}{\sqrt{R}} - \psi. \tag{11}$$

To cancel the secular term, we must therefore introduce the condition that

$$PQ'+x = 0, \quad Q' = -\frac{x}{P}, \tag{12}$$

and now
$$s+x = -\frac{x}{P}\,\frac{r^2}{1-r^2}, \tag{13}$$

$$s = -\frac{x}{P}\,\frac{(1-P)r^2+P}{1-r^2}. \tag{14}$$

In the projection of the catenary on a plane perpendicular to the axis of rotation, if ϕ denotes the radial angle,

$$\cot\phi = \frac{dr}{rd\theta} = \frac{\sqrt{R}}{A}, \tag{15}$$

$$\sin\phi = \frac{A}{r\sqrt{\{(1-r^2)(r^2-b)^2+A^2\}}}, \tag{16}$$

$$p^2 = \frac{A^2}{(1-r^2)(r^2-b)^2+A^2}, \tag{17}$$

or
$$\frac{A^2}{p^2} = (1-r^2)(r^2-b)^2+A^2. \tag{18}$$

The projection will have points of inflexion where p is stationary, corresponding to the maximum of $(1-r^2)(r^2-b)^2$, and therefore when

$$r^2 = \tfrac{1}{3}(b+2), \tag{19}$$

$$\frac{A^2}{p^2} = \tfrac{4}{27}(1-b)^3+A^2. \tag{20}$$

8. To illustrate the preceding theory, begin with the simplest parameter obtained by the bisection of a period, and put

$$v = \tfrac{1}{2}\omega_3. \tag{21}$$

We have to put, for a catenary on a paraboloid,

$$s+x = \frac{Qz}{z+a} = \frac{Qr^2}{r^2+4a^2} = Q\sin^2\omega, \tag{22}$$

in the associated pseudo-elliptic integral

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{-\frac{1}{2}(s+x)+x}{(s+x)\sqrt{S}}\,ds \\ &= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{s}}{2(s+x)} = \tfrac{1}{2}\cos^{-1}\frac{\sqrt{\{4(s+x)^2-s\}}}{2(s+x)}, \end{aligned} \tag{23}$$

in which $$S = 4s(s+x)^2 - s^3, \tag{24}$$

obtained from the general case by putting

$$y = 0. \tag{25}$$

Then, substituting from (22),

$$(z+a)^3 S = 4\{(Q-x)z - xa\}Q^2z^2 - (z+a)\{(Q-x)z-xa\}^2,$$

and this, in consequence of (29), § 3, must assume a form

$$= z(Az-B)^2 - x^2a^3, \tag{26}$$

and thus

$$\begin{aligned} A^2 &= (Q-x)(4Q^2-Q+x), \\ 2AB &= 4Q^2x + (Q-x)(Q-3x), \\ B^2 &= 2Qx - 3x^2, \end{aligned} \tag{27}$$

$$4(Q-x)(4Q^2-Q+x)(2Qx-3x^2) = \{(1+4x)Q^2 - 4Qx + 3x^2\}^2,$$

$$(1-24x+16x^3)Q^4 + 48x^3Q^3 - 6x^2(1+4x)Q^2 + 8x^3Q - 3x^4 = 0, \tag{28}$$

obtainable from the general case of (52), § 4, by putting $y = 0$.

Then, putting

$$Q = \frac{3x}{R+2}, \tag{29}$$

$$R^4 - 6(1-12x)R^2 + 8(1-18x)R - 3(1-12x)^2 = 0, \tag{30}$$

a Jacobian quartic, with

$$\begin{aligned} a &= 1-12x, \quad b = 1-18x, \\ c^3 &= b^2 - a^3 = -108x^2(1-16x). \end{aligned} \tag{31}$$

Writing this equation as a quadratic in x,

$$(R-1)^3(R+3)+72(R-1)^2x-432x^2=0, \tag{32}$$

so that, putting $$12x=(R-1)^2q, \tag{33}$$

$$R+3+6(R-1)q-3(R-1)q^2=0, \tag{34}$$

$$R-1=\frac{4}{3(q-1)^2-4}. \tag{35}$$

Put $$q-1=\tfrac{2}{3}r, \tag{36}$$

$$R-1=\frac{3}{r^2-3}, \tag{37}$$

$$x=\frac{2r+3}{4(r^2-3)^2}, \tag{38}$$

$$Q=\frac{2r+3}{4(r^2-2)(r^2-3)}, \tag{39}$$

$$1-16x=\frac{(r+1)^3(r-3)}{(r^2-3)^2}. \tag{40}$$

It is convenient to put

$$\frac{r-3}{r+1}=p^2 \quad \text{or} \quad -p^2, \tag{41}$$

according as the discriminant

$$\Delta=x^4(1-16x) \tag{42}$$

is positive or negative; and then

$$1-16x=\frac{64p^2}{(p^4-6p^2-3)^2} \quad \text{or} \quad \frac{-64p^2}{(p^4+6p^2-3)^2}; \tag{43}$$

$$x=\frac{(p^2-1)^3(p^2-9)}{16(p^4-6p^2-3)^2} \quad \text{or} \quad \frac{(p^2+1)^3(p^2+9)}{16(p^4+6p^2-3)^2}; \tag{44}$$

$$Q=\frac{(p^2-1)^3(p^2-9)}{8(p^4-6p^2-3)(p^4-10p^2-7)} \quad \text{or} \quad \frac{(p^2+1)^3(p^2+9)}{8(p^4+6p^2-3)(p^4+10p^2-7)}, \tag{45}$$

$$\kappa^2=\left(\frac{p+1}{p-1}\right)^6\left(\frac{p-3}{p+3}\right)^2 \quad \text{or} \quad \frac{(p^4+6p^2-3)^2}{(p^2+1)^3(p^2+9)}=\frac{1}{16x}, \tag{46}$$

$$\kappa'^2=\frac{32p(p^4-6p^2-3)}{(p-1)^6(p+3)^2} \quad \text{or} \quad \frac{64p^2}{(p^2+1)^3(p^2+9)}. \tag{47}$$

The roots of the cubic (B) are given by $s_3 = 0$, and

$$s_1, s_2 = \frac{1-8x \pm \sqrt{(1-16x)}}{8}$$

$$= \left\{ \frac{1 \pm \sqrt{(1-16x)}}{4} \right\}^2, \tag{48}$$

$$= \frac{(p \mp 1)^6 (p \pm 3)^2}{16 (p^4-6p^2-3)^2} \quad \text{or} \quad \frac{(ip \mp 1)^6 (ip \pm 3)^2}{16 (p^4+6p^2-3)^2}; \tag{49}$$

and then $4\gamma^2 = \frac{z_3}{a} = \frac{x}{Q-x} = \frac{p^4-10p^2-7}{(p^2-1)^2}$ or $\frac{p^4+10p^2-7}{(p^2+1)^2}$, (50)

while $(\beta \pm \gamma)^2 = \frac{z_1, z_2}{a} = \frac{p^4-10p^2-7}{4(p \mp 1)^2}$ or $\frac{p^4+10p^2-7}{4(ip \mp 1)^2}$; (51)

so that $\frac{a}{z_1} + \frac{a}{z_2} = \frac{a}{z_3} - 1$, and $\frac{\beta}{\gamma} = p$ or ip. (52)

The constants h and k of the problem are determined by

$$\frac{h}{a} = \frac{B}{A} = \frac{4Q^2x + (Q-x)(Q-3x)}{2(Q-x)(4Q^2-Q+x)}$$

$$= \frac{(p^2+3)(p^4-10p^2-7)}{4(p^2-1)^2}, \tag{53}$$

$$\frac{k^2}{a^2} = \frac{x^2}{A^2} = \frac{x^2}{(Q-x)(4Q^2-Q+x)}$$

$$= \frac{(p^4-10p^2-7)^3}{(p^2-1)^4}. \tag{54}$$

Now we shall find that, in the closed branch of the catenary between z_3 and z_2,

$$\psi = \tfrac{1}{2}(1+\kappa) F\phi - I, \tag{56}$$

where

$$I = \tfrac{1}{2} \sin^{-1} \frac{\sqrt{(p^4-10p^2-7)}}{\sqrt{(p^2-1 \,.\, p^2-9)}} \frac{\sqrt{(z+a \,.\, z-z_3)}}{z}$$

$$= \tfrac{1}{2} \cos^{-1} \frac{4}{\sqrt{(p^2-1 \,.\, p^2-9)}} \frac{\sqrt{(z-z_1 \,.\, z-z_2)}}{z}, \tag{57}$$

and

$$\sin^2 \phi = \frac{z_2+a}{z_2-z_3} \frac{z-z_3}{z+a}, \text{ &c.} \tag{58}$$

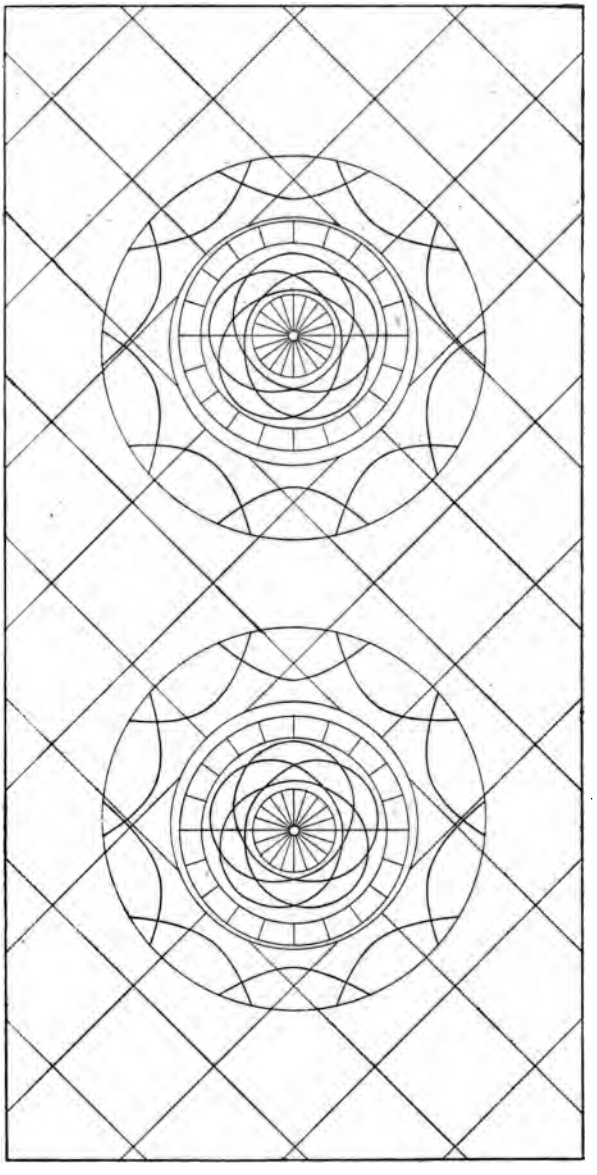
In the open infinite branch, extending from z_1 to infinity,

$$\psi = \tfrac{1}{2}(1+\kappa) F\phi + I, \tag{59}$$

but in this I the $\sin^{-1}$ and $\cos^{-1}$ in (57) must be interchanged; also

$$\sin^2 \phi = \frac{z_2+a}{z_1+a} \frac{z-z_1}{z-z_2}, \text{ &c.} \tag{60}$$

In the stereoscopic diagram, drawn by Mr. T. I. Dewar, we have taken $\kappa = 0{\cdot}77384$, so as to make the apsidal angles 144° and 54°.



Catenary on a Paraboloid.

9. In the associated case of the whirling spherical catenary, we put

$$s+x = \frac{Q'r^2}{1-r^2} = Q' \tan^2\theta, \tag{61}$$

agreeing with the case of the catenary on the paraboloid, if we take

$$r^2 = -\frac{z}{a}. \tag{62}$$

thus utilizing the imaginary part of the catenary, and the negative values of Q, given by the quartic (28), and making

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{-\frac{1}{2}\frac{Qr^2}{1-r^2}+x}{\frac{Qr^2}{1-r^2}\ \frac{A\sqrt{R}}{(1-r^2)^{\frac{1}{2}}}}\ \frac{Q\,dr^2}{(1-r^2)^{\frac{3}{2}}} \\ &= \tfrac{1}{2}\int \frac{-(\frac{1}{2}Q+x)\,r^2+x}{Ar^2\sqrt{(1-r^2)}\sqrt{R}}\,dr^2 \\ &= -\tfrac{1}{2}\,\frac{\frac{1}{2}Q+x}{A}\int \frac{dr^2}{\sqrt{(1-r^2)}\sqrt{R}} + \psi. \end{aligned} \tag{63}$$

If we should try to cancel the secular term by putting

$$\tfrac{1}{2}Q+x = 0, \tag{64}$$

then x must be negative to make Q positive.

Substituting in the expression for S,

$$S\frac{(1-r^2)^2}{x^2} = (1-16x)\,r^4+(1-16x)\,r^4-r^2-1, \tag{65}$$

and it is not possible to construct a real case which shall make this assume the form

$$\frac{r^2\,(r^2-b)^2-A^2}{H^2}. \tag{66}$$

10. For a parameter obtained by the trisection of a period, take

$$v = \tfrac{2}{3}\omega_s, \tag{67}$$

and build up solutions on the pseudo-elliptic integral

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{\frac{1}{3}s+m}{s\sqrt{S}}\,ds \\ &= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} = \tfrac{1}{3}\cos^{-1}\frac{s+m}{2s^{\frac{3}{2}}}, \end{aligned} \tag{68}$$

where

$$S = 4s^3-(s+m)^2. \tag{69}$$

Putting $$s = t^2,$$

$$I(v) = \tfrac{2}{3}\sin^{-1}\frac{\sqrt{(2t^3-t^2-m)}}{2t^{\frac{3}{2}}} = \tfrac{2}{3}\cos^{-1}\frac{\sqrt{(2t^3+t^2+m)}}{2t^{\frac{3}{2}}}, \quad (70)$$

and the associated catenaries on a paraboloid, or on a whirling sphere, are obtained by putting

$$t^2 = s = Q\frac{z}{z+a} = Q\sin^2\omega, \quad t = \sqrt{Q}\sin\omega, \quad (71)$$

or $$t^2 = s = Q'\frac{r^2}{1-r^2} = Q'\tan^2\theta, \quad t = \sqrt{Q'}\tan\theta; \quad (72)$$

and Q, Q' are determined by the condition that

$$\tan\omega_3 = \tan\omega_1 \pm \tan\omega_2, \quad (73)$$

or $$\sin\theta_3 = \sin\theta_1 \pm \sin\theta_2. \quad (74)$$

Then $$\frac{s_1}{s} = \frac{z_1}{z}\frac{z+a}{z_1+a} \quad \text{or} \quad \frac{r_1^2}{r^2}\frac{1-r^2}{1-r_1^2}, \quad (75)$$

$$1-\frac{s_1}{s} = \frac{a}{z}\frac{z-z_1}{z_1+a} \quad \text{or} \quad \frac{1}{r^2}\frac{r^2-r_1^2}{1-r_1^2}, \quad (76)$$

$$\psi+\tfrac{1}{6}qu = I(v) = \tfrac{1}{3}\sin^{-1}\sqrt{\left(1-\frac{s_1}{s}.1-\frac{s_2}{s}.1-\frac{s_3}{s}\right)}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{a^3}{z^3}\ \frac{z-z_1.z-z_2.z-z_3}{z_1+a.z_2+a.z_3+a}\right)}$$

$$\text{or} \quad \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{1}{r^6}\ \frac{r^2-r_1^2.r^2-r_2^2.r^2-r_3^2}{1-r_1^2.1-r_2^2.1-r_3^2}\right)}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left\{\frac{a^3}{z^3}\ \frac{z(z-h)^2-ak^2}{a(a+h)^2+ak^2}\right\}}$$

$$\text{or} \quad \tfrac{1}{3}\sin^{-1}\frac{1}{r^3}\sqrt{\left\{\frac{r^2(r^2-b^2)^2-A^2}{(1-b^2)^2-A^2}\right\}}. \quad (77)$$

Again, since $$\psi+\tfrac{1}{6}u = I(v) = \tfrac{1}{3}\cos^{-1}\frac{s+m}{2s^{\frac{3}{2}}}, \quad (78)$$

therefore it assumes the form

$$\tfrac{1}{3}\cos^{-1}\frac{(Hz-K)\sqrt{(z+a)}}{z^{\frac{3}{2}}\sqrt{\{(a+h)^2+k^2\}}} \quad \text{or} \quad \tfrac{1}{3}\cos^{-1}\frac{(Hr^2-K)\sqrt{(1-r^2)}}{r^3\sqrt{\{(1-b^2)^2-A^2\}}}, \quad (79)$$

where
$$H^2 = 2ah + h^2 + k^2 \quad \text{or} \quad A^2 + 2b^2 - b^4, \tag{80}$$
$$2HK = a(h^2 + k^2) \quad \text{or} \quad -A^2, \tag{81}$$
$$K^2 = a^2k^2 \quad \text{or} \quad A^2 - b^4, \tag{82}$$
and therefore
$$a^2(h^2+k^2)^2 - 4a^2k^2(2ah + h^2 + k^2) = 0,$$
$$(h^2+k^2)(h^2-3k^2) = 8ahk^2, \tag{83}$$
or
$$4(A^2+2b^2-b^4)(A^2-b^4) - A^4 = 0,$$
$$3A^4 + 8A^2b^2(1-b^2) - 8b^6 + 4b^8. \tag{84}$$

Otherwise, the equation (52), § 4, becomes in this case
$$(1+32m)Q^4 + 48m^2Q^3 - 6m^2Q^2 - 8m^3Q - 3m^4 = 0; \tag{85}$$
or, on putting
$$Q = \frac{3m}{R-2}, \tag{86}$$
$$R^4 - 6R^2 - 8(1+54m)R - 3 = 0, \tag{87}$$
so that
$$m = \frac{(R+1)^3(R-3)}{432R}, \tag{88}$$
$$Q = \frac{(R+1)^3(R-3)}{144R(R-2)}, \tag{89}$$
and then, in the catenary on the paraboloid, we find
$$\frac{h}{a} = -\frac{2R(R-2)}{R^2-1}, \tag{90}$$
$$\frac{k^2}{a^2} = -\frac{4R(R-2)^3}{3(R^2-1)^2}. \tag{91}$$

If, in (83), we had put
$$k^2 = h^2p, \tag{92}$$
then
$$\frac{h}{a} = \frac{8p}{(1+p)(1-3p)}, \tag{93}$$
agreeing with (90), when
$$p = -\frac{R-2}{3R}. \tag{94}$$

But there does not appear to be any simple explicit relation of x in terms of R, or R in terms of x.

In the whirling catenary the secular term would be cancelled by putting
$$Q' = -Q = 3m, \tag{95}$$
but this makes $R = 1$, and h and k infinite.

11. Other trisection formulas, corresponding to parameters

$$v = \tfrac{1}{3}\omega_3 \quad \text{or} \quad \tfrac{2}{3}\omega_3, \tag{96}$$

can be obtained by taking (*L.M.S.*, xxv, p. 216)

$$\gamma_6 = 0 \quad \text{or} \quad x = y - y^2, \tag{97}$$

and then

$$\begin{aligned} I(\tfrac{1}{3}\omega_3) &= \tfrac{1}{2}\int \frac{\tfrac{2}{3}(s+y-y^2)-y(1-y)}{s+y-y^2}\,\frac{ds}{\sqrt{S}} \\ &= \tfrac{1}{3}\sin^{-1}\frac{s-\tfrac{1}{2}(1-y)(1-2y)}{(s+y-y^2)^{\frac{1}{2}}}\sqrt{(s-y^2)} \\ &= \tfrac{1}{3}\cos^{-1}\frac{\sqrt{\{s^2-\tfrac{1}{4}(1-y)(1-sy)s+\tfrac{1}{4}y^2(1-y)^2\}}}{(s+y-y^2)^{\frac{1}{2}}} \end{aligned} \tag{98}$$

or

$$\begin{aligned} I(\tfrac{2}{3}\omega_3) &= \tfrac{1}{2}\int \frac{\tfrac{1}{3}(1-3y)s-y^2(1-y)}{s}\,\frac{ds}{\sqrt{S}} \\ &= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} \\ &= \tfrac{1}{3}\cos^{-1}\frac{(1-3y)s-y^2(1-y)}{2s^{\frac{3}{2}}}. \end{aligned} \tag{99}$$

To construct an algebraical catenary on a paraboloid by utilizing the integral (99), put

$$1-3y = 0, \quad y = \tfrac{1}{3}, \quad x = \tfrac{2}{9}. \tag{100}$$

But now

$$S = 4s^3 - \tfrac{4}{7\,2\,9}, \tag{101}$$

and, putting

$$s = \frac{Qz}{z+a}, \tag{102}$$

$$\begin{aligned} (z+a)^3 S &= 4Q^3z^3 - \tfrac{4}{7\,2\,9}(z+a)^3 \\ &= z(Bz-Ca)^2 - \tfrac{4}{7\,2\,9}a^3, \end{aligned} \tag{103}$$

the requisite form, provided that

$$B^2 = 4Q^3 - \tfrac{4}{7\,2\,9}, \quad 2BC = -\tfrac{4}{2\,4\,3}, \quad C^2 = -\tfrac{4}{2\,4\,3}, \tag{104}$$

leading to the value

$$Q = \frac{1}{9\sqrt[3]{4}}; \tag{105}$$

and therefore

$$(z+a)^3 S = -\frac{z}{243}(z+2a)^2 - \tfrac{4}{7\,2\,9}a^3, \tag{106}$$

so that the catenary is imaginary.

In the same way, in attempting to construct an algebraical case of the whirling spherical catenary, by putting, in (98) or (99),

$$s+y-y^2 \quad \text{or} \quad s=\frac{Q'r^2}{1-r^2}, \tag{107}$$

where
$$Q'=-\tfrac{3}{2}y\,(1-y) \quad \text{or} \quad -\frac{3y^2\,(1-y)}{1-3y}, \tag{108}$$

the results obtained are again imaginary.

12. An algebraical case of a catenary on a paraboloid, corresponding to a parameter

$$v=\tfrac{1}{4}\omega_3, \tag{109}$$

can be constructed by taking the pseudo-elliptic integral on p. 228, *L.M.S.*, xxv, and putting

$$z=\tfrac{3}{4}; \tag{110}$$

this makes
$$x=-\tfrac{3}{8}, \quad y=-\tfrac{3}{2},$$

$$S_1=-\tfrac{5}{16}, \quad S_2=\tfrac{3}{32}, \quad S_3=\tfrac{9}{256},$$

$$a=\frac{3^2\times 19}{2^{12}}, \quad b=\frac{3^3\times 215}{2^{18}}, \quad c=\frac{3^5\sqrt{2}}{2^{12}}; \tag{111}$$

whence the equation for Q can be constructed, when we put

$$s-\tfrac{3}{8}=\frac{Qz}{z+a}=\frac{Qr^2}{r^2+4a^2}, \tag{112}$$

and now the catenary is given by putting this expression for s in

$$\begin{aligned}\psi&=\tfrac{1}{4}\sin^{-1}\frac{\sqrt{(s-\frac{9}{16})}}{4\,(s-\frac{3}{8})^2}\\&=\tfrac{1}{4}\cos^{-1}\frac{(s-\frac{5}{8})\sqrt{(16s^2-4s+\frac{9}{4})}}{4\,(s-\frac{3}{8})^2}.\end{aligned} \tag{113}$$

13. To form the corresponding algebraical whirling spherical catenary, we start with the pseudo-elliptic integral (*L.M.S.*, xxv, p. 228)

$$\begin{aligned}I(\tfrac{1}{4}\omega_3)&=\int\frac{\frac{1}{4}\dfrac{3-4z}{1-z}\,(s+z-2z^2)-z+2z^2}{s+z-2z^2}\,\frac{ds}{\sqrt{S}}\\&=\tfrac{1}{4}\cos^{-1}\frac{(s-1+2z-2z^2)\sqrt{\left\{4s^2-\dfrac{(1-2z)^4}{(1-z)^2}\,s+\dfrac{z^2\,(1-2z)^4}{(1-z)^2}\right\}}}{2\,(s+z-2z^2)^2}\end{aligned}$$

$$= \tfrac{1}{4}\sin^{-1}\frac{\dfrac{(3-4z)\,s-(1-2z)^2}{1-z}\sqrt{(s-z^2)}}{2\,(s+z-2z^2)^2}, \tag{114}$$

and put
$$s+z-2z^2 = \frac{Q'r^2}{1-r^2} \tag{115}$$

with
$$Q' = -\frac{4z\,(1-z)\,(1-2z)}{3-4z}, \tag{116}$$

thereby cancelling the secular term.

Then
$$s-z^2 = -\frac{z\,(1-z)}{3-4z}\cdot\frac{(1-4z)\,r^2+3-4z}{1-r^2}, \tag{117}$$

so that
$$r_1^2 = -\frac{3-4z}{1-4z}. \tag{118}$$

Also
$$4s^2-\frac{(1-2z)^4}{(1-z)^2}s+\frac{z^2(1-2z)^4}{(1-z)^2} = \frac{z\,(1-2z)^2}{(1-z)\,(3-4z)^2\,(1-r^2)^2}$$
$$\times\left\{-(3-8z+8z^2)(1-8z+8z^2)\,r^4-2\,(3-4z)(1-8z+8z^2)\,r^2+(3-4z)^2\right\}, \tag{119}$$

so that
$$r_2^2+r_3^2 = -2\frac{3-4z}{3-8z+8z^2}, \tag{120}$$

$$r_2^2 r_3^2 = -\frac{(3-4z)^2}{(3-8z+8z^2)\,(1-8z+8z^2)}, \tag{121}$$

$$r_1^2-r_2^2-r_3^2 = -\frac{(3-4z)\,(1+8z^2)}{(1-4z)\,(3-8z+8z^2)}. \tag{122}$$

Then, from the conditions
$$(r_1^2-r_2^2-r_3^2)^2 = 4r_2^2r_3^2, \tag{123}$$

we obtain the equation
$$(1+8z^2)^2\,(1-8z+8z^2)+4\,(1-4z)^2\,(3-8z+8z^2) = 0 \tag{124}$$

or
$$512z^6-512z^5+704z^4-896z^3+504z^2-136z+13 = 0, \tag{125}$$

a sextic equation for the determination of z.

Putting $z = \frac{1}{4}p$, this becomes
$$p^6-4p^5+22p^4-112p^3+252p^2-272p+104 = 0, \tag{126}$$

the roots of which are
$$p = 2{\cdot}77764,\ 0{\cdot}7604,\ 1{\cdot}22961 \pm 0{\cdot}92181i,\ 0{\cdot}90861 \pm 4{\cdot}45039i, \tag{127}$$

as calculated by Mr. T. I. Dewar.

Choosing the root $p = 2{\cdot}77764$, this makes

$$z = 0{\cdot}6944, \tag{128}$$

$$r_1^2 = 0{\cdot}1251, \quad r_1 = \sin 20^\circ\, 42', \tag{129}$$

$$Q' = 0{\cdot}1386, \tag{130}$$

$$1-8z+8z^2 = -0{\cdot}68, \tag{131}$$

$$3-8z+8z^2 = 1{\cdot}32, \tag{132}$$

$$b = \tfrac{1}{2}(r_1^2+r_2^2+r_3^2) = -0{\cdot}1096, \tag{133}$$

$$A^2 = r_1^2 r_2^2 r_3^2 = 0{\cdot}008. \tag{134}$$

At the point of inflexion on the projection,

$$r^2 = \tfrac{1}{3}(2+b) = 0{\cdot}63, \quad r = \sin 52^\circ\, 32'. \tag{135}$$

Substituting for s in terms of r^2 from (117) in (114), we shall obtain equations of the form

$$\psi = \tfrac{1}{4}\sin^{-1}\frac{(Hr^2+H_1)\sqrt{(1-r^2.r^2-r_1^2)}}{r^4}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(Kr^2+K_1)\sqrt{(r^2-r_2^2.r^2-r_3^2)}}{r^2}, \tag{136}$$

$$H = \frac{(3-4z.4z-1)^{\frac{1}{2}}}{32\,(2z-1)\,(z-z^2)^{\frac{1}{2}}}, \quad H_1 = \frac{(3-4z)^{\frac{1}{2}}\,(4z-1)^{\frac{1}{2}}}{32\,(2z-1)\,(z+z^2)^{\frac{1}{2}}}, \tag{137}$$

$$K = \frac{(3-8z+8z^2)^{\frac{1}{2}}\,(-1+8z-8z^2)^{\frac{1}{2}}}{32\,(2z-1)\,(z-z^2)^{\frac{1}{2}}\,(3-4z)^{\frac{1}{2}}}, \quad K_1 = -\frac{(3-8z+8z^2)^{\frac{1}{2}}\,(-1+8z-8z^2)^{\frac{1}{2}}}{32\,(2z-1)\,(z-z^2.3-4z)^{\frac{1}{2}}}, \tag{138}$$

giving the form of the whirling spherical catenary shown in the stereoscopic diagram annexed, drawn by Mr. T. I. Dewar.

14. In the case of quinquisection, with a parameter

$$v = \tfrac{2}{5}\omega_3, \tag{139}$$

the relation $\gamma_5 = 0$ is satisfied by $y = x$, and now

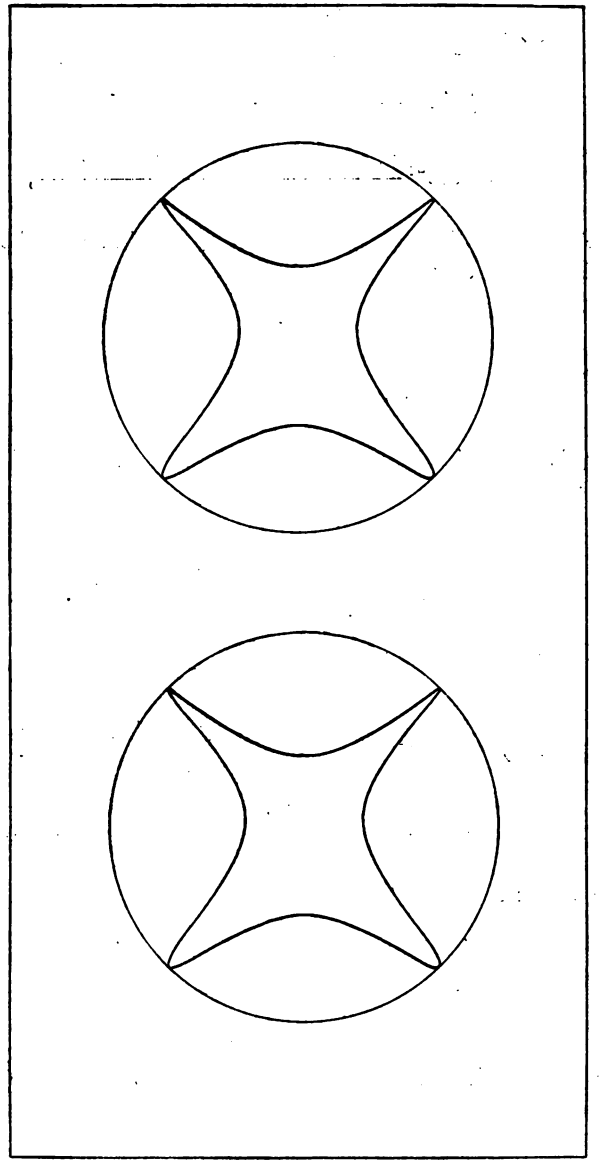
$$S_1 = \tfrac{1}{4}(x^2+6x+1), \quad S_2 = \tfrac{1}{2}(x^2+x), \quad S_3 = \tfrac{1}{4}x^2; \tag{140}$$

and

$$a = \tfrac{1}{16}x^2(x^2-10x+1), \tag{141}$$

$$b = -\tfrac{1}{64}x^3(x^3+39x^2-15x+1), \tag{142}$$

$$c^3 = \frac{27x^9}{1024}(x^2+11x-1). \tag{143}$$



Catenary on a Whirling Sphere.

and therefore leads to the equation for x,

$$32x^3+27x-7=0,$$

$$x=\frac{-27\pm5\sqrt{(65)}}{64}, \tag{164}$$

of which the positive value makes Q' negative, and must therefore be rejected.

Taking, then, $$x+3=\frac{165-5\sqrt{(65)}}{64}, \tag{165}$$

$$Q'=\frac{5\sqrt{(65)}+27}{33-\sqrt{(65)}}$$

$$=\frac{3\{7\sqrt5+3\sqrt{(13)}\}^2}{1024}, \tag{166}$$

$$q=\frac{1}{\sqrt{Q}}=\frac{32}{\sqrt3\{7\sqrt5+3\sqrt{(13)}\}}$$

$$=\frac{7\sqrt5-3\sqrt{(13)}}{4\sqrt3}; \tag{167}$$

and, now, with $$t=\frac{\tan\theta}{q}, \tag{168}$$

the equation of the catenary can be written in either of the forms

$$(\tan\theta)^{\frac12}\sin\tfrac52\psi=(\tan\theta+q)\sqrt{\{\tfrac12\tan^3\theta-\tfrac14(1-x)q\tan^2\theta}$$
$$-\tfrac12xq^2\tan\theta+\tfrac14xq^3\}, \tag{169}$$

$$(\tan\theta)^{\frac12}\cos\tfrac52\psi=(\tan\theta-q)\sqrt{\{\tfrac12\tan^3\theta+\tfrac14(1-x)q\tan^2\theta}$$
$$-\tfrac12xq^2\tan\theta-\tfrac14xq^3\}. \tag{170}$$

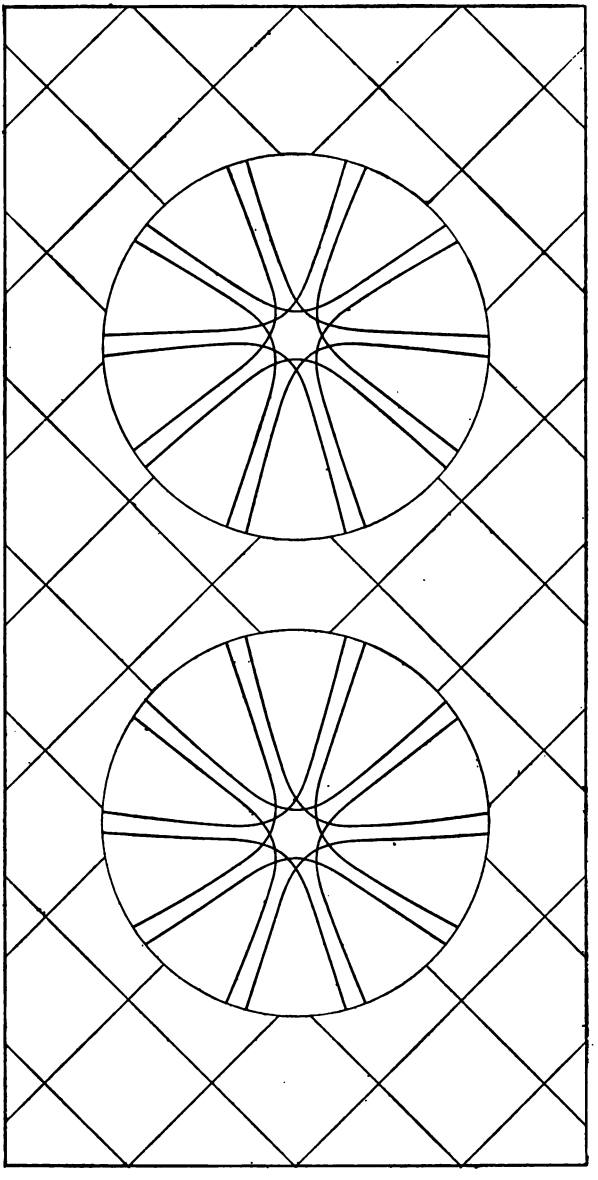
So also the equation of the algebraical catenary on the paraboloid may be written

$$(\sin\omega)^{\frac12}\sin\tfrac52\psi=(\sin\omega+q)\sqrt{(\tfrac12\sin^3\omega-q\sin^2\omega+\tfrac32q^2\sin\omega-\tfrac34q^3)}, \tag{171}$$

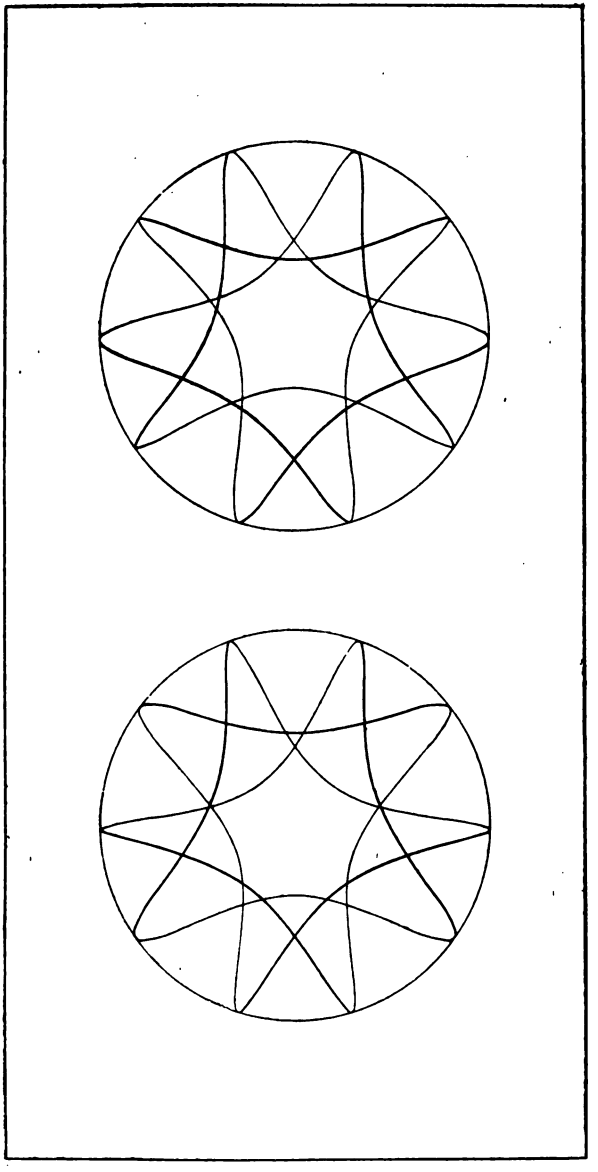
$$(\sin\omega)^{\frac12}\cos\tfrac52\psi=(\sin\omega-q)\sqrt{(\tfrac12\sin^3\omega+q\sin^2\omega+\tfrac32q^2\sin\omega+\tfrac34q^3)}, \tag{172}$$

where $$q=\frac{1}{\sqrt{(4.5)}}=0{\cdot}4714.$$

These two catenaries are represented in the annexed stereoscopic diagrams, drawn by Mr. T. I. Dewar.



Catenary on a Paraboloid.


Catenary on a Whirling Sphere.

16. With a parameter $v = \frac{2}{7}\omega_3$, (173)

the associated pseudo-elliptic integral (*L.M.S.*, XXVI, p. 225)

$$I(v) = \tfrac{1}{2}\int \frac{\frac{1}{7}(5-z-z^2)(s+x)-x}{(s+x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{7}\sin^{-1}\frac{s^3+Cs+D}{2(s+x)^{\frac{1}{2}}}\sqrt{S}, \quad \&c., \tag{174}$$

in which $x = z(1-z)^2, \quad y = z(1-z),$ (175)

and, again putting $s+x = t^2$, the result can be expressed by

$$I(v) = \tfrac{2}{7}\sin^{-1}\frac{(t^3-t-1+z)\sqrt{\{2t^3-(1-z-z^2)t^2-2z(1-z)t+z(1-z)^2\}}}{2t^{\frac{1}{2}}}$$

$$= \tfrac{2}{7}\cos^{-1}\frac{(t^3+t-1+z)\sqrt{\{2t^3+(1-z-z^2)t^2-2z(1-z)t-z(1-z)^2\}}}{2t^{\frac{1}{2}}}, \tag{176}$$

and, to obtain the corresponding catenaries, we put

$$t = \sqrt{Q}\sin\omega \quad \text{or} \quad \sqrt{Q'}\tan\theta, \tag{177}$$

and determine Q from the condition that S assumes the form

$$M\,\frac{r^2(Br^2-C)^2-E}{(r^2+4a^2)^3 \text{ or } (1-r^2)^3}. \tag{178}$$

To make the catenary on the paraboloid algebraical, put

$$5-z-z^2 = 0, \quad z = \frac{-1\pm\sqrt{(21)}}{2}. \tag{179}$$

But, to obtain an algebraical whirling spherical catenary, put

$$Q' = \frac{-7z(1-z)^2}{5-z-z^2}, \tag{180}$$

and now it will be found that

$$\left.\begin{aligned} B^2 &= 4(1-8z+5z^2+z^3)(5-19z+11z^2-9z^3)\\ 2BC &= -8(5-z-z^2)(2-3z)(1-8z+5z^2+z^3)\\ C^2 &= (5-z-z^2)^2(1-17z+11z^2)\end{aligned}\right\}, \tag{181}$$

thus leading to the equation

$$4(2-3z)^2(1-8z+5z^2+z^3)-(1-17z+11z^2)(5-19z+11z^2-9z^3) = 0$$

or $$(5z-1)(27z^4-23z^3-26z^2+17z-11) = 0. \tag{182}$$

The value $z = \frac{1}{5}$ makes Q' negative; of the other four roots

$$+1{\cdot}376, \quad -1{\cdot}03, \quad \text{and} \quad 0{\cdot}253\pm 0{\cdot}464i, \tag{183}$$

the negative root will make Q' positive, and give a real case.

The Trajectory of a Particle sliding on a smooth Paraboloid.

17. The path of a particle on a smooth paraboloid whose axis is vertical is of a similar analytical character to the catenary curve. (Züge, *Grunert's Archiv*, 70, 1884.)

The equations for the conservation of momentum and energy are

$$r^2 \frac{d\psi}{dt} = H, \tag{1}$$

$$\tfrac{1}{2}\frac{dz^2}{dt^2} + \tfrac{1}{2}\frac{dr^2}{dt^2} + \tfrac{1}{2}r^2\frac{d\psi^2}{dt^2} = g\,(z \sim h)\,; \tag{2}$$

with
$$r^2 = 4az, \tag{3}$$

$$\frac{dr}{dt} = \sqrt{\frac{a}{z}}\,\frac{dz}{dt} \tag{4}$$

Thence
$$\left(1+\frac{a}{z}\right)\frac{dz^2}{dt^2} = 2g\,(z \sim h) - \frac{H^2}{4az}, \tag{5}$$

or
$$\frac{dz^2}{dt^2} = \frac{2gz\,(z \sim h) - \dfrac{H^2}{4a}}{z+a}$$

$$= 2g\,\frac{z\,(z \sim h) - \frac{1}{4}k_2^2}{z+a}\,; \tag{6}$$

putting
$$H^2 = 2gak^2 \quad \text{and} \quad 4az\frac{d\psi}{dt} = H = \sqrt{(2gak^2)}, \tag{8}$$

so that, dividing (8) by (6),

$$\frac{d\psi}{dz} = \tfrac{1}{2}\frac{k}{\sqrt{a}}\,\frac{\sqrt{(z+a)}}{z\sqrt{\{4z\,(z \sim h) - k^2\}}}$$

$$= \tfrac{1}{2}\frac{k}{\sqrt{a}}\,\frac{z+a}{z\sqrt{Z}}, \tag{9}$$

where
$$Z = (z+a)\,\{4z\,(z \sim h) - k^2\}. \tag{10}$$

Putting
$$g = 2an^2, \tag{11}$$

$$n\frac{dt}{dz} = \frac{z+a}{\sqrt{(aZ)}}, \tag{12}$$

so that the time is given by Legendre's $E\phi$ and $F\phi$.

Taking r^2 as independent variable,

$$\psi = \tfrac{1}{2}k\int \frac{(r^2+4a^2)\,dr^2}{r^2\sqrt{R}}, \tag{13}$$

where
$$R = (r^2+4a^2)\,\{r^2\,(r^2 \sim 4ah) - 4a^2k^2\}, \tag{14}$$

so that the projection of the catenary is of herpolhode character.

(i.) When the vertex of the paraboloid is upwards,

$$Z = (z+a)\{4z(z-h)-k^2\}$$
$$= 4(z-z_3)(z-z_2)(z-z_1), \quad (15)$$

where $z_1 = \frac{1}{2}h+\frac{1}{2}\sqrt{(h^2+k^2)}$, (16)

$z_2 = \frac{1}{2}h-\frac{1}{2}\sqrt{(h^2+k^2)}$, or $-a$, a negative quantity, (17)

$z_3 = -a$, or $\frac{1}{2}h-\frac{1}{2}\sqrt{(h^2+k^2)}$; (18)

so that $\infty > z > z_1 > 0 > z_2 > z_3 > -\infty$,

and, putting $z = M^2(s-\sigma) = \wp u-\wp v$, (19)

then $u = p\omega_1$, (20)

for real parts of the path of the particle, while

$$v = \omega_1+f\omega_3. \quad (21)$$

Then $z+a = M^2(s-s_3)$ or $M^2(s-s_2)$, (22)

$a = M^2(\sigma-s_3)$ or $M^2(\sigma-s_2)$, (23)

$z_1+z_{2,3} = h = M^2(s_1+s_{2,3}-2\sigma)$, (24)

$z_1-z_{2,3} = \sqrt{(h^2+k^2)} = M^2(s_1-s_{2,3})$, (25)

$$k\sqrt{a} = M^3\sqrt{(-\Sigma)}, \quad (26)$$

$$\frac{k}{\sqrt{a}} = M\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}, \quad (27)$$

$$\sqrt{Z} = M^3\sqrt{S}, \quad (28)$$

$$dz = M^2 ds, \quad (29)$$

$$\frac{dz}{\sqrt{Z}} = \frac{1}{M}\frac{ds}{\sqrt{S}}; \quad (30)$$

and, introducing these substitutions into (9),

$$\psi = \tfrac{1}{2}k\int_{z_1}^{z}\frac{z+a}{z\sqrt{(aZ)}}dz$$
$$= \tfrac{1}{2}\int_{s_1}^{s}\left\{\frac{\sqrt{(-\Sigma)}}{s-\sigma}+\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}\right\}\frac{ds}{\sqrt{S}}$$
$$= I(v)-\tfrac{1}{2}\left\{P(v)-\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}\right\}\int\frac{ds}{\sqrt{S}}$$
$$= I(v)-\tfrac{1}{2}MP(v-\omega_{2,3})\int\frac{dz}{\sqrt{Z}}$$
$$= I(v)-\tfrac{1}{2}MP(v-\omega_{2,3})\frac{F\phi}{\sqrt{(z_1-z_3)}}, \quad (31)$$

in which $$\sin^2\phi = \frac{z-z_1}{z-z_2}, \quad \&c., \tag{32}$$

and now, in (12),

$$ndt = \frac{z-z_{2,3}}{\sqrt{(aZ)}}\,dz$$

$$= \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{\Delta\phi\,d\phi}{\kappa'\cos^2\phi} \quad \text{or} \quad \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{\kappa'^2 d\phi}{\cos^2\phi\,\Delta\phi}, \tag{33}$$

so that (Legendre, I, p. 257)

$$nt = \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{1}{\kappa'}\,(\tan\phi\,\Delta\phi + F\phi - E\phi)$$

or

$$\sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,(\tan\phi\,\Delta\phi + \kappa'^2 F\phi - E\phi). \tag{34}$$

18. As the secular term cannot be cancelled, it will be sufficient to consider the single special case of a parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3; \tag{35}$$

and now, with $$z+a = M^2(s-s_3), \quad z = M^2(s-\sigma), \tag{36}$$

we shall find (*L.M.S.*, xxv, p. 212)

$$I = \tfrac{1}{2}\cos^{-1}\frac{\sqrt{(hz+ah)}}{z} = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z^2-hz-ah)}}{z}, \tag{37}$$

with $$k^2 = 4ah = \frac{4a^2}{c+c^2}, \tag{38}$$

$$z_1,\ z_2 = \tfrac{1}{2}h \pm \sqrt{(\tfrac{1}{4}h^2+ah)} = \frac{a}{c} \quad \text{or} \quad -\frac{a}{1+c}. \tag{39}$$

Differentiating (37), $$\frac{dI}{dz} = \sqrt{h}\,\frac{\frac{1}{2}z+a}{z\sqrt{Z}}$$

$$= \frac{d\psi}{dz} - \tfrac{1}{2}\frac{\sqrt{h}}{\sqrt{Z}}. \tag{40}$$

Also $$\kappa = \sqrt{\left(\frac{z_2-z_3}{z_1-z_3}\right)} = \frac{\sqrt{(4a+h)}-\sqrt{h}}{\sqrt{(4a+h)}+\sqrt{h}}, \tag{41}$$

and $$\tfrac{1}{2}\int_{z_1}^{z}\frac{\sqrt{h}\,dz}{\sqrt{Z}} = \tfrac{1}{2}\sqrt{\left(\frac{h}{z_1-z_3}\right)}\,F\phi$$

$$= \frac{\sqrt{h}F\phi}{\sqrt{(4a+h)}+\sqrt{h}} = \tfrac{1}{2}(1-\kappa)\,F\phi, \tag{42}$$

with $$\sin^2\phi = \frac{z-z_1}{z-z_2} = \frac{z-\frac{a}{c}}{z+\frac{a}{1+c}} = \frac{z-a\frac{1-\kappa}{\kappa}}{z+a(1-\kappa)}, \quad \&c., \tag{43}$$

so that the equation of the trajectory is

$$\psi = I + \tfrac{1}{2}(1-\kappa)F\varphi. \tag{44}$$

Expressed in terms of r^2, this may be written in either of the forms

$$r^2\cos\{2\psi-(1-\kappa)F\varphi\} = \surd\{4ah(r^2+4a^2)\}, \tag{45}$$

$$r^2\sin\{2\psi-(1-\kappa)F\varphi\} = \surd(r^4-4ahr^2-16a^3h)$$

$$= \sqrt{\left[\left(r^2-4a^2\frac{1-\kappa}{\kappa}\right)\{r^2+4a^2(1-\kappa)\}\right]}; \tag{46}$$

and, while r^2 ranges from $4a^2\frac{1-\kappa}{\kappa}$ to infinity, the angle ψ increases from zero to

$$\Psi = \frac{\pi}{4}\left\{(1-\kappa)\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right\}. \tag{47}$$

With the same parameter, but with

$$z+a = M^2(s-s_2) \quad = M^2(s-c^2), \tag{48}$$

$$z = M^2(s-\sigma) \quad = M^2(s-c-c^2), \tag{49}$$

$$a = M^2c, \tag{50}$$

$$h = z_1+z_3 \quad = M^2(1-c^2), \tag{51}$$

$$\surd(h^2+k^2) = z_1-z_3 \quad = M^2(1+c^2), \tag{52}$$

$$z_1 = M^2(1+c) \quad = a\frac{1+c}{c}, \tag{53}$$

$$z_2 = -M^2c \quad = -a, \tag{54}$$

$$z_3 = -M^2(c+c^2) = -a(1+c), \tag{55}$$

$$I = \tfrac{1}{2}\sin^{-1}\frac{\surd(z-z_1\,.\,z-z_3)}{z} \quad = \tfrac{1}{2}\cos^{-1}\sqrt{\frac{a}{c}}\frac{\surd(z-z_3)}{z}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\left(z-a\frac{1+c}{c}\,.\,z+a\right)}}{z} = \tfrac{1}{2}\cos^{-1}\sqrt{\frac{a}{c}}\frac{\surd\{z+a(1+c)\}}{z}, \tag{56}$$

and, by differentiation, we find

$$\frac{dI}{dz} = \frac{d\psi}{dz} - \frac{1+2c}{2\surd c}\frac{\surd a}{\surd Z}. \tag{57}$$

To reduce the secular term to Legendre's standard form, put

$$\sin^2\phi = \frac{z-a\frac{1+c}{c}}{z+a}, \quad \cos^2\phi = \frac{a\frac{1+2c}{c}}{z+a}, \quad \Delta^2\phi = \frac{1+2c}{(1+c)^2}\,\frac{z+a(1+c)}{z+a}, \tag{58}$$

and then
$$\frac{2+c}{2\sqrt{c}}\,\frac{\sqrt{a}\,dz}{\sqrt{Z}} = \frac{1+2c}{2+2c}\,\frac{d\phi}{\Delta\phi} = \tfrac{1}{2}(1+\kappa)\,\frac{d\phi}{\Delta\phi}, \tag{59}$$

so that
$$\psi - \tfrac{1}{2}(1+\kappa)\,F\phi = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z-z_1\,.\,z-z_2)}}{z}, \quad \&c., \tag{60}$$

and the equation of the projection of the catenary may be written, as before:

$$r^2\sin\left\{2\psi-(1+\kappa)\,F\phi\right\} = \sqrt{\left\{\left(r^2-\frac{4a^2}{\kappa}\right)(r^2+4a^2)\right\}}, \tag{61}$$

$$r^2\cos\left\{2\psi-(1+\kappa)\,F\phi\right\} = \frac{2a}{\sqrt{c}}\sqrt{\left(r^2+\frac{4a^2}{1-\kappa}\right)}, \tag{62}$$

and, while r ranges from $2a/\sqrt{\kappa}$ to infinity, ψ increases from zero to

$$\Psi = \frac{\pi}{4}\left\{(1+\kappa)\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right\}. \tag{63}$$

Thus, for instance, with $c=1$, $h=0$, $k=4a$, $\kappa=\frac{1}{2}=\sin 30°$.

19. To find the pressure on the surface, we notice that the cosine of the angle between the normal and the axis of the paraboloid is

$$\sqrt{\left(\frac{a}{z+a}\right)}, \tag{64}$$

so that, denoting the pressure between the particle and the outside surface by R in dynes, the mass of the particle by m in grammes, and resolving vertically,

$$m\frac{d^2z}{dt^2} = mg - R\sqrt{\left(\frac{a}{z+a}\right)}. \tag{65}$$

But, since
$$\frac{dz^2}{dt^2} = 2g\,\frac{z^2-hz-\frac{1}{4}k^2}{z+a}, \tag{66}$$

$$\frac{d^2z}{dt^2} = g\,\frac{z^2+2az-ah+\frac{1}{4}k^2}{(z+a)^2}, \tag{67}$$

so that
$$R\sqrt{\left(\frac{a}{z+a}\right)} = mg\,\frac{a^2+ah-\frac{1}{4}k^2}{(z+a)^2}, \tag{68}$$

$$\frac{R}{mg} = \frac{a^2+ah-\frac{1}{4}k^2}{\sqrt{a}\,(z+a)^{\frac{3}{2}}}. \tag{69}$$

Then, if
$$a^2+ah = \tfrac{1}{4}k^2, \quad z_2 = z_3 = -a, \tag{70}$$

$$R = 0, \tag{71}$$

and the particle moves freely in space in a parabolic trajectory.

But, in this case, equation (9) reduces to

$$\frac{d\psi}{dz} = \tfrac{1}{2}\,\frac{\frac{k}{2\sqrt{a}}}{z\sqrt{\left(z-\frac{k^2}{4a}\right)}}. \tag{72}$$

and integrating,

$$\psi = \cos^{-1}\frac{k}{2\sqrt{(az)}} = \cos^{-1}\frac{k}{r}, \tag{73}$$

or
$$r\cos\psi = k, \tag{74}$$

so that the projection of the path on a horizontal plane is a straight line, a verification.

20. When the vertex of the paraboloid is downwards, the particle must now move on the interior of the surface, and

$$\begin{aligned} Z &= (z+a)\left\{-4z\,(z-h)-k^2\right\} \\ &= -4\,(z-z_1)\,(z-z_2)\,(z-z_3), \end{aligned} \tag{75}$$

where
$$z_1 = -a, \tag{76}$$

$$z_2 = \tfrac{1}{2}h-\tfrac{1}{2}\sqrt{(h^2-k^2)}, \tag{77}$$

$$z_3 = \tfrac{1}{2}h+\tfrac{1}{2}\sqrt{(h^2-k^2)}, \tag{78}$$

and
$$\infty > z_3 > z > z_2 > 0 > z_1 > -\infty,$$

so that the parameter
$$v = \omega_1 + f\omega_3, \tag{79}$$

and we put
$$z = M^2(\sigma-s) = \wp v - \wp u, \tag{80}$$

$$z+a = M^2(s_1-s), \tag{81}$$

$$a = M^2(s_1-\sigma), \tag{82}$$

$$z_3+z_2 = h = M^2(2\sigma-s_2-s_3), \tag{83}$$

$$z_3-z_2 = \sqrt{(h^2-k^2)} = M^2(s_2-s_3), \tag{84}$$

and the equation of the trajectory may be written

$$r^2 \cos\{2\psi-(1+\kappa)F\phi\} = \surd(r^4+4a^2.r^2-4\kappa a^2) \qquad (111)$$

or
$$r^2 \sin\{2\psi-(1+\kappa)F\phi\} = 2a\surd(1-\kappa)\sqrt{\left(\frac{4\kappa}{1-\kappa}\kappa^2-r^2\right)}$$
$$= 2a\surd\{4\kappa a^2-(1-\kappa)r^2\}. \qquad (112)$$

Thus, as r diminishes from $2a\sqrt{\left(\frac{\kappa}{1-\kappa}\right)}$ to $2a\surd\kappa$, the angle Ψ increases by

$$\Psi = \tfrac{1}{4}\pi\left\{(1+\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \qquad (113)$$

We can make the apsidal angle Ψ very nearly $\frac{4}{5}\pi$ or 144°, by trial and error, by taking, as in Fig. 1, p. 599,

$$\kappa = 0{\cdot}77384 = \sin 50°\,42',$$

and
$$\frac{r_2}{2a} = \surd\kappa = 0{\cdot}8796, \quad \frac{r_3}{2a} = \sqrt{\left(\frac{\kappa}{1-\kappa}\right)} = 1{\cdot}849. \qquad (114)$$

We thus obtain a trajectory which very nearly closes upon itself.

The pressure R is again obtained by resolving in the vertical direction

$$m\frac{d^2z}{dt^2} = R\sqrt{\left(\frac{a}{z+a}\right)}-mg, \qquad (115)$$

and
$$\frac{d^2z}{dt^2} = g\,\frac{-z^2-2az+ah+\frac{1}{4}k^2}{(z+a)^2}, \qquad (116)$$

so that
$$\frac{R}{mg} = \frac{a^2+ah+\frac{1}{4}k^2}{\surd a\,(z+a)^{\frac{3}{2}}}, \qquad (117)$$

so that R does not change sign.

The Catenary on a Vertical Cone.

22. Taking the semi-vertical angle of the cone as α, then, along a generator,

$$x = z\tan\alpha, \qquad (1)$$

and the general equation (5), in § 1, for the curve formed by a chain wrapped on a vertical cone of revolution becomes

$$\frac{d\psi}{dz} = \frac{A\sec\alpha}{z\tan\alpha\surd\{z^2(z-h)^2\tan^2\alpha-A^2\}}$$

or
$$\sin\alpha\frac{d\psi}{dz} = \frac{k^2}{z\surd\{z^2(z-h)^2-k^4\}}, \qquad (2)$$

putting
$$A = k^2\tan\alpha. \qquad (3)$$

If r, θ denote the polar coordinates when the surface of the cone is developed into a plane, we can put

$$\psi \sin \alpha = \theta \quad \text{and} \quad r \cos \alpha = z, \tag{4}$$

so that

$$\frac{d\theta}{dr} = \frac{b^2}{r\sqrt{\{r^2(r-2a)^2 - b^4\}}} = \frac{b^2}{r\sqrt{R}}, \tag{5}$$

involving only one parameter b/a; and

$$b = k \sec \alpha, \quad 2a = h \sec \alpha, \tag{6}$$

$$R = r^2(r-2a)^2 - b^4 = (r^2 - 2ar + b^2)(r^2 - 2ar - b^2). \tag{7}$$

This proves that in the developed catenary (Routh, *Analytical Statics*, I, p. 361)

$$\frac{1}{p} = \alpha + \beta r, \tag{8}$$

where p denotes the perpendicular from the origin on the tangent, and α, β are constants; for, ϕ denoting the *radial* angle,

$$r\frac{d\theta}{dr} = \tan \phi = \frac{b^2}{\sqrt{\{r^2(r-2a)^2 - b^4\}}}, \tag{9}$$

$$\sin \phi = \frac{b^2}{r(r-2a)}, \tag{10}$$

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi\right) = \sqrt{\left(\frac{r^2 - 2ar - b^2}{r^2 - 2ar + b^2}\right)}, \tag{11}$$

and

$$p = r \sin \phi = \frac{b^2}{r-2a}, \tag{12}$$

$$p(r-2a) = b^2. \tag{13}$$

Thus $h = 0$, or $a = 0$, gives

$$pr = b^2, \tag{14}$$

a rectangular hyperbola.

23. Put

$$r - a = x; \tag{15}$$

then

$$\theta = \int \frac{b^2 dx}{(x+a)\sqrt{X}}, \tag{16}$$

where

$$X = (x^2 - a^2 + b^2)(x^2 - a^2 - b^2). \tag{17}$$

We can now employ the Jacobian notation.

When $a^2 - b^2$ is positive, $= \beta^2$, suppose, and

$$a^2 + b^2 = a^2, \quad a^2 = \tfrac{1}{2}(a^2 + \beta^2);$$

then

$$X = x^2 - a^2 . x^2 - \beta^2. \tag{18}$$

On the open branch of the curve we put

$$x = \frac{a}{\operatorname{sn} u}, \tag{19}$$

and on the closed branch $x = \beta \operatorname{sn} u,$ (20)

with $\kappa = \frac{\beta}{a},$ (21)

so that increasing u by $2K'i$ changes from the closed to the open branch of the curve.

We put $a = \beta \operatorname{sn} v,$ (22)

and then $\operatorname{cn} v = \frac{ib}{\beta}, \quad \operatorname{dn} v = \frac{b}{a},$ (23)

and $v = K + fK'i;$ (24)

so that we can write equation (16)

$$\theta = \int \frac{b^2 x\,dx}{(x^2-a^2)\sqrt{X}} - \int \frac{ab^2\,dx}{(x^2-a^2)\sqrt{X}}$$

$$= \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2-b^2}{x^2-a^2}\right)} + \int \frac{\frac{i \operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}}{\operatorname{sn}^2 v - \operatorname{sn}^2 u}\,du, \tag{25}$$

involving a Jacobian Elliptic Integral of the Third Kind, in a standard form.

24. It follows, from the preceding relations (23), that

$$\operatorname{sn}^2 (1-f)\,K'i = -1; \tag{26}$$

and, to the complementary modulus K',

$$\operatorname{tn}^2 (1-f)\,K' = 1, \quad (1-f)\,K' = F(\tfrac{1}{4}\pi), \tag{27}$$

so that, in the pseudo-elliptic applications the results are restricted to special numerical cases.

In equation (25),

$$\sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2-b^2}{x^2-a^2}\right)} = \cos^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2+b^2}{x^2-a^2}\right)}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{b^2}{x^2-a^2} = \tfrac{1}{2}\cos^{-1}\frac{b^2}{r(r-2a)}$$

$$= \tfrac{1}{2}(\tfrac{1}{2}\pi - \phi); \tag{28}$$

from (10), § 22, so that (25) may be written

$$\theta+\tfrac{1}{2}\phi-\tfrac{1}{4}\pi = \int \frac{i\,\dfrac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}}{\operatorname{sn}^2 v-\operatorname{sn}^2 u}\,du. \tag{29}$$

Thus in the rectangular hyperbola

$$\theta = \tfrac{1}{4}\pi-\tfrac{1}{2}\phi. \tag{30}$$

24. Also, if s' denotes the length of the arc

$$\frac{ds'}{dz} = \frac{ds}{r\,d\psi}\,\frac{r^2 d\psi}{dz} = \frac{z-h}{A}\,\frac{r^2 A \sec a}{z\tan^2 a\sqrt{Z}}$$

$$= \frac{z\,(z-h)\sec a}{\sqrt{Z}}, \tag{31}$$

$$\frac{ds'}{dr} = \frac{r\,(r-2a)}{\sqrt{R}}, \tag{32}$$

$$\frac{ds'}{dx} = \frac{x^2-a^2}{\sqrt{X}}, \tag{33}$$

so that s' is given by Elliptic Integrals of the First and Second Kind.

25. Denoting by σ' the value of s corresponding to

$$u = (1-f)\,K'i, \tag{34}$$

then

$$\operatorname{sn}^2 (1-f)\,K'i = \frac{s_1-s_3}{\sigma'-s_3}, \tag{35}$$

so that, from (26),

$$s_1-s_3 = s_3-\sigma'. \tag{36}$$

Thus, for $f = \frac{1}{2}$,

$$\sigma' = -c-c^2, \quad s_1 = (1+c)^2, \quad s_2 = c^2, \quad s_3 = 0; \tag{37}$$

so that

$$c = \infty, \quad b = 0, \tag{38}$$

which must be rejected, as representing a plane curve, a straight line along a generator.

If

$$f = \tfrac{1}{3}, \quad \sigma' = 0,$$

$$s_1 = (1-c)^2, \quad s_2 = c^2, \quad s_3 = (c-c^2)^2, \tag{39}$$

so that

$$1-2c^2 = 0, \quad c = \tfrac{1}{2}\sqrt{2} = 0{\cdot}707, \tag{40}$$

which must be rejected, as greater than 0·5.

If

$$f = \tfrac{2}{3}, \quad \sigma' = -2c+2c^2, \tag{41}$$

leading to

$$2c^2-4c+1 = 0,$$

$$c = 1-\tfrac{1}{2}\sqrt{2} = 0{\cdot}293; \tag{42}$$

$$\frac{\beta^2}{a^2} = \kappa^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{(\sqrt{2}-1)^2}{2\sqrt{2}-1} = \frac{4\sqrt{2}-5}{7} = \frac{a^2-b^2}{a^2+b^2}, \tag{43}$$

$$\kappa'^2 = \frac{12-4\sqrt{2}}{7}, \tag{44}$$

$$\frac{b^2}{a^2} = 2(\sqrt{2}-1), \tag{45}$$

$$\frac{a^2}{a^2} = 2\sqrt{2}-1, \tag{46}$$

$$\frac{\beta^2}{a^2} = (\sqrt{2}-1)^2. \tag{47}$$

The equation of the catenary can now be written

$$\theta = \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-(2\sqrt{2}-1)\,a^2}{x^2-a^2}\right)} + I + \tfrac{1}{3}\sqrt{2}\int\frac{a\,dx}{\sqrt{X}}, \tag{48}$$

where
$$I = \tfrac{1}{3}\cos^{-1}\frac{\left\{x^2-(\sqrt{2}+1)\,a^2\right\}\sqrt{\left\{x^2-(\sqrt{2}-1)^2\,a^2\right\}}}{(x^2-a^2)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{2}\,\frac{ax\sqrt{\left\{x^2-(2\sqrt{2}-1)\,a^2\right\}}}{(x^2-a^2)^{\frac{3}{2}}}, \tag{49}$$

and
$$\tfrac{1}{3}\sqrt{2}\int\frac{a\,dx}{\sqrt{X}} = \tfrac{1}{3}\sqrt{2}\,\frac{a}{a}F\phi, \tag{50}$$

where
$$\sin\phi = \frac{x}{\beta} \text{ in the limited branch,}$$

$$\sin\phi = \frac{a}{x} \text{ in the unlimited branches.}$$

26. When a^2-b^2 is negative, $= -\beta^2$, and $a^2+b^2 = a^2$,

$$X = x^2+\beta^2.x^2-a^2, \tag{51}$$

and we put
$$x = \frac{a}{\mathrm{cn}\,u}, \quad \kappa = \frac{\beta}{\sqrt{(a^2+\beta^2)}}, \tag{52}$$

and the curve consists of a single open branch.

To reduce the integral to the standard form, we put

$$\mathrm{cn}\,u = \frac{a}{x}, \quad \mathrm{cn}\,v = \frac{a}{a}, \tag{53}$$

so that
$$u = pK, \quad v = fK'i, \tag{54}$$

and then
$$\mathrm{sn}^2 v = -\frac{b^2}{a^2}, \quad \mathrm{dn}^2 v = \frac{a^2}{2a^2}, \tag{55}$$

so that
$$\mathrm{cn}^2 v - 2\,\mathrm{dn}^2 v = 0, \tag{56}$$

$$\mathrm{cn}\{(1-f)K', \kappa'\} = \frac{\kappa}{\kappa'}. \tag{57}$$

Then, from (25),

$$\begin{aligned}\theta &= \int \frac{b^2 x\,dx}{(x^2-a^2)\sqrt{X}} - \int \frac{ab^2 dx}{(x^2-a^2)\sqrt{X}} \\ &= \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2}{x^2-\alpha^2}\right)} - \frac{ab^2}{a^2\sqrt{(a^2+\beta^2)}}\int \frac{du}{\mathrm{nc}^2 u - \mathrm{nc}^2 v} \\ &= \tfrac{1}{2}(\tfrac{1}{2}\pi-\phi) - \frac{i\,\mathrm{sn}\,v\,\mathrm{dn}\,v}{\mathrm{cn}^2 v}\int \frac{du}{\mathrm{nc}^2 u - \mathrm{nc}^2 v},\end{aligned} \tag{58}$$

depending upon a Jacobian elliptic integral, in a standard form.

To change from the Jacobian form to that given in (A), § 3, we put

a^2-b^2 positive	a^2-b^2 negative
$a^2 = \frac{1}{2}(\alpha^2+\beta^2)$	$a^2 = \frac{1}{2}(\alpha^2-\beta^2)$
$b^2 = \frac{1}{2}(\alpha^2-\beta^2)$	$b^2 = \frac{1}{2}(\alpha^2+\beta^2)$
$x^2 = m^2(s-s_3)$	$x^2 = m^2(s-s_2)$
$x^2-\alpha^2 = m^2(s-s_1)$	$x^2-\alpha^2 = m^2(s-s_1)$
$x^2-\beta^2 = m^2(s-s_2)$	$x^2+\beta^2 = m^2(s-s_3)$
$x^2-a^2 = m^2(s-\sigma)$	$x^2-a^2 = m^2(s-\sigma)$

(59)

so that

$\sigma = \frac{1}{2}(s_1+s_2)$	$\sigma = \frac{1}{2}(s_1+s_3)$
$\alpha^2 = m^2(s_1-s_3)$	$\alpha^2 = m^2(s_1-s_2)$
$\beta^2 = m^2(s_2-s_3)$	$\beta^2 = m^2(s_2-s_3)$
$m^3 S = x\sqrt{(x^2-\alpha^2.x^2-\beta^2)}$	$m^3 S = \sqrt{(x^2-\alpha^2.x^2+\beta^2)}$

Now
$$\frac{ds}{\sqrt{S}} = m\frac{dx}{\sqrt{X}}, \tag{60}$$

and
$$m^3\sqrt{(-\Sigma)} = 2ab^2, \tag{61}$$

$$\tfrac{1}{2}\frac{m\sqrt{(-\Sigma)}}{s-\sigma} = \frac{ab^2}{x^2-a^2}, \tag{62}$$

so that
$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi - \tfrac{1}{2}\int \frac{\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}; \tag{63}$$

and, therefore, from (A), § 3,

$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi - I + \tfrac{1}{2}mP\int \frac{dx}{\sqrt{X}}. \tag{64}$$

27. Try $$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{65}$$

with $$\sigma = \tfrac{1}{2}(s_1 + s_2); \tag{66}$$

then (*L.M.S.*, xxv, p. 212)

$$c + c^2 = \tfrac{1}{2}(1+c)^2 + \tfrac{1}{2}c^2, \tag{67}$$

giving $c = \infty$, which must be rejected.

But, with $$\sigma = \tfrac{1}{2}(s_1 + s_3), \tag{68}$$

$$c + c^2 = \tfrac{1}{2}(1+c)^2, \tag{69}$$

and $$c = 1, \quad \kappa = \tfrac{1}{2}, \tag{70}$$

$$\frac{a^2}{\beta^2} = \frac{s_1 - s_2}{s_2 - s_3} = 3, \quad a^2 = 3\beta^2 = 3a^2, \tag{71}$$

$$b^2 = 2a^2, \tag{72}$$

and we find that $$I = \tfrac{1}{2}\sin^{-1}\frac{x\sqrt{(x^2-3a^2)}}{x^2-a^2}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{a\sqrt{(x^2+a^2)}}{x^2-a^2}, \tag{73}$$

with $$P = 1, \quad m = a, \text{ in (64).} \tag{74}$$

Try $$v = \omega_1 + \tfrac{1}{3}\omega_3, \tag{75}$$

with $$\sigma = \tfrac{1}{2}(s_1 + s_2); \tag{76}$$

then (*L.M.S.*, xxv, p. 218)

$$2c(1-c)^2 = \tfrac{1}{2}(1 - 2c + 2c^2), \tag{77}$$

$$(2c-1)(2c^2 - 4c + 1) = 0, \tag{78}$$

of which the root $$c = 1 - \tfrac{1}{2}\sqrt{2} = 0{\cdot}293 \tag{79}$$

must be taken, as before, in (42), § 25. (80)

With $$\sigma = \tfrac{1}{2}(s_1 + s_3), \tag{81}$$

we have $$(c-1)^2(c^2 - 4c + 1) = 0, \tag{82}$$

of which the root $$c = 2 - \sqrt{3} \tag{83}$$

makes $$\kappa = \sin 15°. \tag{84}$$

With $$v = \omega_1 + \tfrac{1}{4}\omega_3, \tag{85}$$

$$\sigma = \tfrac{1}{2}(s_1 + s_2) \text{ or } \tfrac{1}{2}(s_1 + s_3), \tag{86}$$

we take (*L.M.S.*, xxv, p. 226)

$$-z(1-2z) = \frac{(1-2z)^4}{8(1-z)^2}, \tag{87}$$

reducing to $$(2z-1)(4z^3-2z-1) = 0, \tag{88}$$

so that $$z = \tfrac{1}{2}, \quad -\sin 18°, \quad \sin 54°. \tag{89}$$

The root $z = -\sin 18°$ makes

$$1-8z+8z^3 = 2+\sqrt{5}, \tag{90}$$

and must therefore be chosen.

Having determined in this manner the numerical value of the modulus, &c., the corresponding value of I as a function of x is calculated, and the secular term $\int \frac{dx}{\sqrt{X}}$ is reduced to Legendre's form $F\phi$ by the appropriate substitutions; so that the equation of the catenary is expressed in functions which have been tabulated numerically.

28. Although the theory of the catenary on a cone is now analytically complete, still we shall find it more convenient, because of the occurrence of the term $\frac{1}{2}(\frac{1}{2}\pi-\phi)$, to adopt another mode of reduction of the integral, equivalent to Landen's quadric transformation performed on the first substitution.

Suppose R then to be resolved into linear factors, so that

$$R = r-r_0.r-r_3.r-r_2.r-r_1, \tag{1}$$

where $$\infty > r > r_0 > 2a > r_3 > r > r_2 > 0 > r_1 > r > -\infty. \tag{2}$$

To agree with the notation employed in the *Applications of Elliptic Functions*, p. 154, we replace r_0, r_3, r_2, r_1 by α, β, γ, δ; and put

$$\wp u - e_1 = M^2(s-s_1) = \tfrac{1}{4}(\alpha-\beta)(\alpha-\gamma)\frac{r-\delta}{r-\alpha}, \tag{3}$$

$$\wp u - e_2 = M^2(s-s_2) = \tfrac{1}{4}(\alpha-\delta)(\alpha-\beta)\frac{r-\gamma}{r-\alpha}, \tag{4}$$

$$\wp u - e_3 = M^2(s-s_3) = \tfrac{1}{4}(\alpha-\gamma)(\alpha-\delta)\frac{r-\beta}{r-\alpha}, \tag{5}$$

$$\kappa^2 = \frac{e_2-e_3}{e_1-e_3} = \frac{s_2-s_3}{s_1-s_3} = \frac{\beta-\gamma.\alpha-\delta}{\alpha-\gamma.\beta-\delta}, \tag{6}$$

$$\kappa'^2 = \frac{e_1-e_2}{e_1-e_3} = \frac{s_1-s_2}{s_1-s_3} = \frac{\alpha-\beta.\gamma-\delta}{\alpha-\gamma.\beta-\delta}. \tag{7}$$

We can put $a=1$, without loss of generality; also we put

$$b^4 = 1-e^2, \tag{8}$$

$$R = r^2(r-2)^2-1+e^2, \tag{9}$$

so that
$$a = r_0 = 1+\sqrt{\left(\frac{1+e}{2}\right)}+\sqrt{\left(\frac{1-e}{2}\right)}, \tag{10}$$

$$\beta = r_3 = 1+\sqrt{\left(\frac{1+e}{2}\right)}-\sqrt{\left(\frac{1-e}{2}\right)}, \tag{11}$$

$$\gamma = r_2 = 1-\sqrt{\left(\frac{1+e}{2}\right)}+\sqrt{\left(\frac{1-e}{2}\right)}, \tag{12}$$

$$\delta = r_1 = 1-\sqrt{\left(\frac{1+e}{2}\right)}-\sqrt{\left(\frac{1-e}{2}\right)}, \tag{13}$$

and
$$a-\gamma = \beta-\delta, \quad a-\beta = \gamma-\delta, \quad \kappa' = \frac{a-\beta}{a-\gamma}; \tag{14}$$

and this makes
$$\kappa^2 = \frac{2e}{1+e}, \quad \kappa'^2 = \frac{1-e}{1+e}. \tag{15}$$

But, denoting by λ, λ' the moduli introduced in the original substitution in § 23,

$$\lambda = \frac{\beta}{a} = \sqrt{\left(\frac{1-b^2}{1+b^2}\right)} = \sqrt{\left\{\frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}\right\}}$$

$$= \frac{\sqrt{(1+e)}-\sqrt{(1-e)}}{\sqrt{(1+e)}+\sqrt{(1-e)}} = \frac{1-\kappa'}{1+\kappa'}, \tag{16}$$

$$\lambda' = \frac{2\sqrt{\kappa'}}{1+\kappa'}, \tag{17}$$

as in one of Landen's transformations.

The elliptic argument

$$u = \int \frac{dr}{\sqrt{R}} = \frac{2F\varphi}{\sqrt{(a-\gamma.\beta-\delta)}} = \sqrt{\left(\frac{2}{1+e}\right)} F\varphi = \sqrt{(2-\kappa^2)}\, F\varphi, \tag{18}$$

and, in the region of the limited branch of the catenary

$$r_3 > r > r_2, \tag{19}$$

$$F\varphi = \mathrm{sn}^{-1}\sqrt{\left(\frac{\beta-\delta.r-\gamma}{\beta-\gamma.r-\delta}\right)} = \&c. \tag{20}$$

or
$$\sin^2\phi = \frac{\beta-\delta.r-\gamma}{\beta-\gamma.r-\delta}, \ \&c.; \tag{21}$$

while, in the two regions $\infty > r > r_0$ and $r_1 > r > -\infty$, (22)

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{\beta-\delta.r-a}{a-\delta.r-\beta}\right)}, \text{ \&c.,} \tag{23}$$

$$\sin^2\phi = \frac{\beta-\delta.r-a}{a-\delta.r-\beta}, \text{ \&c.} \tag{24}$$

Calculating the invariants g_2 and g_3 of the quartic R in (9), then

$$3g_2 = 1+3e^2, \tag{25}$$

$$27g_3 = 1-9e^2, \tag{26}$$

so that the discriminating cubic

$$4x^3-g_2x-g_3 = 0 \tag{27}$$

breaks up into the factors

$$(x-\tfrac{1}{3})\left\{(2x+\tfrac{1}{3})^2-e^2\right\} = 0, \tag{28}$$

and thus, arranged in descending order,

$$e_1 = \tfrac{1}{3}, \quad e_2 = -\tfrac{1}{6}+\tfrac{1}{2}e, \quad e_3 = -\tfrac{1}{6}-\tfrac{1}{2}e. \tag{29}$$

Calculating the Hessian H of the quartic R,

$$H = -\tfrac{1}{3}R-(1-e^2)(r-1)^2, \tag{30}$$

and, employing Hermite's formula,

$$\wp 2u = -\frac{H}{R} = \tfrac{1}{3}+\frac{(1-e^2)(r-1)^2}{R}, \tag{31}$$

$$\wp 2u-e_1 = \frac{(1-e^2)(r-1)^2}{R}. \tag{32}$$

Since $r = \infty$, and $r = 1$, make

$$\wp 2u-e_1 = 0; \tag{33}$$

it follows that, if w_1 and w_2 denote the corresponding arguments,

$$w_1 = \tfrac{1}{2}\omega_1, \quad w_2 = \omega_3+\tfrac{1}{2}\omega_1. \tag{34}$$

Thus, if A denotes the sectorial area of the developed catenary,

$$dA = \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}b^2\frac{r\,dr}{\sqrt{R}}, \tag{35}$$

so that A is always given by a pseudo-elliptic integral of the third kind; in fact

$$A = \tfrac{1}{2}b^2\,\mathrm{ch}^{-1}\frac{\sqrt{(b^2+2r-r^2)}}{b\sqrt{2}}-\tfrac{1}{2}b^2u, \tag{36}$$

$$= \tfrac{1}{2}b^2\,\mathrm{sh}^{-1}\frac{\sqrt{(-b^2+2r-r^2)}}{b\sqrt{2}}-\tfrac{1}{2}b^2u, \tag{37}$$

for the limited branch; but

$$A = \tfrac{1}{2}b^2 \begin{matrix}\mathrm{ch}^{-1}\\ \mathrm{sh}^{-1}\end{matrix} \frac{\sqrt{(r^2-2r+b^2)}}{b\sqrt{2}} + \tfrac{1}{2}b^2u, \tag{38}$$

$$= \tfrac{1}{2}b^2 \begin{matrix}\mathrm{sh}^{-1}\\ \mathrm{ch}^{-1}\end{matrix} \frac{\sqrt{(r^2-2r-b^2)}}{b\sqrt{2}} + \tfrac{1}{2}b^2u, \tag{39}$$

for the unlimited branches; both forms being included in

$$A = \tfrac{1}{4}b^2 \mathrm{sh}^{-1} \frac{\sqrt{R}}{b^2} \mp \tfrac{1}{2}b^2u. \tag{40}$$

If v_1, v_2 denote the elliptic arguments corresponding to $r = 0$ and $r = 2$, then

$$\wp 2v_1 = \wp 2v_2 = -\tfrac{2}{3}, \tag{41}$$

so that we may put

$$v_1 = \omega_1 + f\omega_3, \quad v_2 = f\omega_3, \tag{42}$$

f denoting a fraction; also

$$i\wp' 2v_1 = i\wp' 2v_3 = \sqrt{(1-e^2)} = b^2. \tag{43}$$

Also

$$\wp'' 2v_1 = 6\wp^2 - \tfrac{1}{2}g_2$$

$$= \tfrac{5}{2} - \tfrac{1}{2}e^2, \tag{44}$$

so that, from the formula

$$\wp 4v + 2\wp 2v = \tfrac{1}{4}\left(\frac{\wp'' 2v}{\wp' 2v}\right)^2, \tag{45}$$

$$\wp 4v_1 = -\frac{1}{1-e^2} + \tfrac{5}{6} - \frac{1-e^2}{16}. \tag{46}$$

Thus, for instance, with

$$v_1 = \omega_1 + \tfrac{1}{4}\omega_3, \tag{47}$$

$$4v_1 = 4\omega_1 + \omega_3, \tag{48}$$

$$\wp 4v_1 = e_3 = -\tfrac{1}{6} - \tfrac{1}{2}e, \tag{49}$$

leading to the equation

$$(e^2-1)^2 + 8(e+2)(e^2-1) + 16 = 0 \tag{50}$$

or

$$(e^2+4e-1)^2 = 0, \tag{51}$$

$$e = \sqrt{5}-2, \tag{52}$$

$$b^4 = 4(\sqrt{5}-2). \tag{53}$$

The other root

$$e = -\sqrt{5}-2 \tag{54}$$

would correspond to the case when R has four imaginary roots.

Making use of Weierstrass's formula,

$$\wp(v_1 \pm v_2) = \frac{\frac{8}{3} - b^4 \mp b^4}{8}$$

$$= \tfrac{1}{12} + \tfrac{1}{4}e^2 \text{ or } \tfrac{1}{3}; \tag{55}$$

agreeing in making $v_1 - v_2 = \omega_1$, (56)

and putting $v_1 + v_2 = v$, (57)

then $12\wp v = 1 + 3e^2$, (58)

while $12g_2 = 4 + 12e^2$

$$= (1+3e^2)^2 + 24\frac{(1-e^2)(1+3e^2)}{8}, \tag{59}$$

$$216g_3 = 8 - 72e^2$$

$$= -(1+3e^2)^3,$$

$$1728\Delta = (12g_2)^3 - (216g_3)^2 = 1728e^3(1-e^2)^2. \tag{60}$$

Forming the invariants of the cubic S in (B), § 3, we can make

$$\frac{12g_2}{M^4} = \{(y+1)^2 + 4x\}^2 - 24x(y+1), \tag{61}$$

$$\frac{216g_3}{M^6} = \{(y+1)^2 + 4x\}^3 - 36x(y+1)\{(y+1)^2 + 4x\} + 216x^3 \tag{62}$$

(*L.M.S.*, XXVII, p. 129), so that, on comparison,

$$-M^2\{(y+1)^2 + 4x\} = 1 + 3e^2, \tag{63}$$

$$-M^4x(y+1) = \tfrac{1}{8}(1-e^2)(1+3e^2), \tag{64}$$

$$16M^6x^3 = (1-e^2)^3; \tag{65}$$

and therefore, from (64) and (65),

$$4M^2(y+1)^2 = \frac{(1+3e^2)^2}{1-e^2}; \tag{66}$$

and, from (63),

$$4M^2x = -\tfrac{1}{4}\frac{(1+3e^2)^2}{1-e^2} - 1 - 3e^2$$

$$= -\frac{(5-e^2)(1+3e^2)}{4(1-e^2)}, \tag{67}$$

$$M^2 = \frac{16(1-e^2)^5}{(5-e^2)^2(1+3e^2)^2}, \tag{68}$$

$$x = -\frac{(5-e^2)^3(1+3e^2)^3}{256(1-e^2)^6}, \tag{69}$$

$$y+1 = \frac{(5-e^2)(1+3e^2)^3}{8(1-e^2)^3}, \tag{70}$$

$$y = -\frac{(3+e^2)(1-18e^2+e^4)}{8(1-e^2)^3}; \tag{71}$$

whence an equation for e^2 can be formed, when we put Halphen's

$$\gamma_n = 0. \tag{72}$$

29. If we turn to the values of Halphen's x and y in terms of a and m, as given in the *L.M.S.*, XXVII, p. 449, we shall find that

$$a = \frac{3+e^2}{4(1-e^2)}, \tag{73}$$

$$m = \frac{(5-e^2)(1+3e^2)}{4(1-e^2)(3+e^2)}, \tag{74}$$

$$a-m = \frac{1-e^2}{3+e^2}, \tag{75}$$

and thence $4a(a-m) = 1$, (76)

$$\frac{a}{a-m} = \frac{(3+e^2)^2}{4(1-e^2)^2}, \tag{77}$$

$$ma = \frac{(5-e^2)(1+3e^2)}{16(1-e^2)^2}, \tag{78}$$

$$s_\gamma = \frac{(5-e^2)^2(1+3e^2)^2(3+e^2)^2}{256(1-e^2)^6}, \tag{79}$$

and $s_\gamma = s_1$, always.

Also, with $s_\alpha = s_2, \quad s_\beta = s_3,$

$$(s_1-s_2)(s_1-s_3) = \frac{(5-e^2)^4(1+3e^2)^4}{1024(1-e^2)^9}, \tag{80}$$

$$(s_2-s_3)^2 = \frac{e^2(5-e^2)^4(1+3e^2)^4}{256(1-e^2)^{10}}, \tag{81}$$

$$(s_1-s_2+s_1-s_3)^2 = \frac{(5-e^2)^4(1+3e^2)^4}{256(1-e^2)^{10}}, \tag{82}$$

$$s_1-s_2,\ s_1-s_3 = \frac{(5-e^2)^2(1+3e^2)^2(1\mp e)}{32(1-e^2)^{10}}, \tag{83}$$

$$s_2,\ s_3 = \frac{(5-e^2)^2(1+3e^2)^2(1\pm 4e-e^2)^2}{256(1-e^2)^6}. \tag{84}$$

Also
$$\kappa^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{2c}{1+c}, \tag{85}$$

$$\kappa'^2 = \frac{s_1 - s_2}{s_1 - s_3} = \frac{1-c}{1+c}, \tag{86}$$

as before, thus serving as a verification.

Having fixed upon some pseudo-elliptic form of the integral $I(v_1)$ in (A), we shall obtain an equation for e, giving it a certain numerical value; and then we must express the variable s in terms of r by means of the relations in (3), (4), (5) of § 28.

We can also employ the formula

$$\wp u = \tfrac{1}{24} R''(r_0) + \tfrac{1}{4} \frac{R'(r_0)}{r - r_0}, \tag{87}$$

where
$$R(r) = r^4 - 2r^3 + r^2 + 0 - b^4, \tag{88}$$

$$R'(r) = 4r^3 - 6r^2 + 2r, \tag{89}$$

$$R''(r) = 12r^2 - 12r + 2, \tag{90}$$

and now
$$\wp u - \wp v_1 = M^2 (s - \sigma) = \tfrac{1}{2} (2r_0^2 - 3r_0 + 1) \frac{r}{r - r_0}. \tag{91}$$

Another simple relation is

$$\frac{s - s_1}{\sqrt{(s_1 - s_2 . s_1 - s_3)}} = \sqrt{\left(\frac{\alpha - \beta . \alpha - \gamma}{\beta - \delta . \gamma - \delta}\right)} \frac{r - r_1}{r - r_0} = \frac{r - r_1}{r - r_0}, \tag{92}$$

because
$$\alpha + \delta = \beta + \gamma, \quad \alpha - \beta = \gamma - \delta, \quad \alpha - \gamma = \beta - \delta, \tag{93}$$

and then
$$\frac{\sqrt{(s_1 - s_2 . s_1 - s_3)}}{s - s\tfrac{1}{2}\omega_1} = \frac{r - r_0}{r_0 - r_1}, \tag{94}$$

because
$$s\tfrac{1}{2}\omega_1 = s_1 + \sqrt{(s_1 - s_2 . s_1 - s_3)}. \tag{95}$$

Putting
$$r = 0, \quad u = v_1, \quad s = \sigma,$$

$$\frac{\sqrt{(s_1 - s_2 . s_1 - s_3)}}{\sigma - s\tfrac{1}{2}\omega_1} = \frac{-r_0}{r_0 - r_1}, \tag{96}$$

$$\frac{\sigma - s\tfrac{1}{2}\omega_1}{s - s\tfrac{1}{2}\omega_1} = 1 - \frac{r}{r_0}, \tag{97}$$

so that
$$\frac{s - \sigma}{s - s\tfrac{1}{2}\omega_1} = \frac{r}{r_0}, \tag{98}$$

$$\frac{s_a - \sigma}{s_a - s\tfrac{1}{2}\omega_1} = \frac{r_a}{r_0}. \tag{99}$$

The relation between the roots

$$r_0 r_1 + r_2 r_3 = 0 \tag{100}$$

thus leads to

$$\frac{s_1 - \sigma}{s_1 - s\frac{1}{2}\omega_1} + \frac{s_2 - \sigma \,.\, s_3 - \sigma}{s_2 - s\frac{1}{2}\omega_1 \,.\, s_3 - s\frac{1}{2}\omega_1} = 0, \tag{101}$$

an equation which may be employed to settle the numerical value of a parameter.

From (3), (4), (5) of § 28,

$$-\wp' u = M^3 \sqrt{S} = \tfrac{1}{4}(r_0 - r_1 \,.\, r_0 - r_2 \,.\, r_0 - r_3) \frac{\sqrt{R}}{(r - r_0)^2}$$

$$= \tfrac{1}{4} R'(r_0) \frac{\sqrt{R}}{(r - r_0)^2}, \tag{102}$$

and

$$M^3 ds = \tfrac{1}{4} R'(r_0) \frac{-dr}{(r - r_0)^2}; \tag{103}$$

therefore

$$\frac{ds}{\sqrt{S}} = -M \frac{dr}{\sqrt{R}}. \tag{104}$$

Also

$$M^3 \sqrt{(-\Sigma)} = \tfrac{1}{4} R'(R_0) \frac{b^2}{r_0^2}, \tag{105}$$

$$\frac{M\sqrt{(-\Sigma)}}{s - \sigma} = b^2 \frac{r - r_0}{r_0 r}, \tag{106}$$

so that

$$d\theta = \frac{b^2 dr}{r\sqrt{R}} = -\frac{r_0}{r - r_0} \frac{\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}}$$

$$= -\frac{s - s\frac{1}{2}\omega_1}{s\frac{1}{2}\omega_1 - \sigma} \frac{\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}}$$

$$= -\frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1 - \sigma} \frac{ds}{\sqrt{S}} + \frac{\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}}$$

$$= 2dI(v_1) - \left\{ P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1 - \sigma} \right\} \frac{ds}{\sqrt{S}}. \tag{107}$$

$$\theta = 2I(v_1) + M \left\{ P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1 - \sigma} \right\} \int \frac{dr}{\sqrt{R}}$$

$$= 2I(v_1) + M \left\{ P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1 - \sigma} \right\} u, \tag{108}$$

and it is the coefficient of the secular term u which is apt to be very baffling; so, to make sure of it in the applications, having constructed the pseudo-elliptic term $I(v_1)$, a differentiation will be employed for the verification.

It will not be necessary, and it would occupy too much space, to go through all the details by which $I(v_1)$ is converted from being a function of s, as worked out in the paper on "Pseudo-Elliptic Integrals," *L.M.S.*, XXV, into a function of r, and in which r_0, r_1, r_2, r_3 have certain numerical values depending upon the solution of an associated equation; so now we proceed to the simplest cases.

Having drawn some branches of the curve on a sheet of paper, we can bend the paper into a conical shape, and, as the vertical angle of the cone is at our disposal, we can, if we like, choose it so as to make the branches close and overlap each other, after an assigned number of convolutions, and thus have a figure suitable for stereoscopic representation.

The stereoscopic diagram on p. 638, drawn by Mr. Dewar, shows the catenary on a cone given by equation (147) in § 30, when the closed branch is made to have an apsidal angle of 180°, so as to form a single loop.

30. With $$v_1 = \omega_1 + \tfrac{1}{2}\omega_3, \quad v_2 = \tfrac{1}{2}\omega_3, \quad v = \omega_1 + \omega_3, \tag{109}$$

$$\wp v = e_2 \quad \text{or} \quad \tfrac{1}{12} + \tfrac{1}{4}e^2 = -\tfrac{1}{6} + \tfrac{1}{2}e, \tag{110}$$

so that $$e = 1, \quad b = 0, \tag{111}$$

and the catenary degenerates into a generating line of the cone.

Next, with $$v_1 = \omega_1 + \tfrac{1}{3}\omega_3, \quad v_2 = \tfrac{1}{3}\omega_3, \quad v = \omega_1 + \tfrac{2}{3}\omega_3, \tag{112}$$

and $$\wp 3v = e_1, \tag{113}$$

so that [*L.M.S.*, XXVII, (326), p. 450]

$$m = 1, \tag{114}$$

and therefore, from (74),

$$e^4 + 22e^2 - 7 = 0, \tag{115}$$

$$e^2 = 8\sqrt{2} - 11 = (\sqrt{2}-1)^2(2\sqrt{2}-1), \tag{116}$$

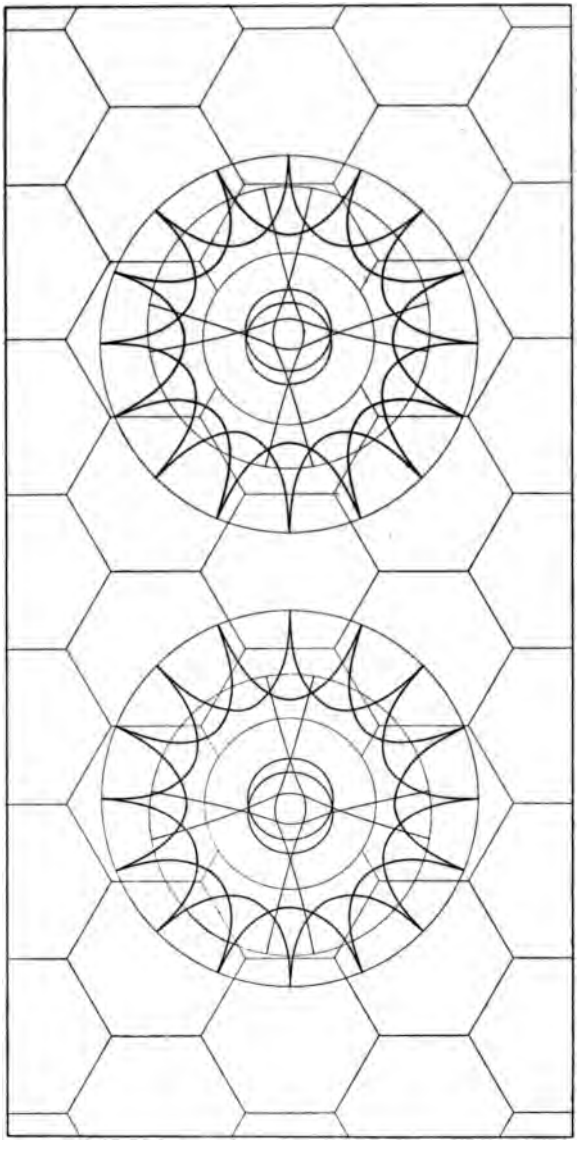
$$b^4 = 12 - 8\sqrt{2}, \quad b^2 = 2\sqrt{2} - 2, \tag{117}$$

$$\kappa = \tfrac{1}{2}\sqrt[4]{(2\sqrt{2}-1)}\left\{\sqrt{(2\sqrt{2}-1)} - \sqrt{2} + 1\right\}. \tag{118}$$

The c employed in *L.M.S.*, XXV, p. 217, is now given by

$$(1-c)^2 = \frac{s_3^2}{s_2^2} = \left(\frac{1-4e-e^2}{1+4e-e^2}\right)^2, \tag{119}$$

$$1 - c = \pm\frac{1-4e-e^2}{1+4e-e^2}, \tag{120}$$



Catenary on a Cone.

in which the negative sign must be taken; so that

$$c = 1 + \frac{1-4e-e^2}{1+4e-e^2} = 2\,\frac{1-e^2}{1+4e-e^2} = \frac{\sqrt{(2\sqrt{2}-1)} - \sqrt{2}+1}{2}. \quad (121)$$

In the catenary curve the corresponding value of $I(v_1)$ for the infinite branches is found to be given by

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r-r_0 . r-r_1 . r-r_3)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{(r+r_2)\sqrt{(r-r_2)}}{r^{\frac{3}{2}}}, \quad (122)$$

with $r_0 = 1+\sqrt{(2\sqrt{2}-1)}$, (123)

$r_3 = \sqrt{2}$, (124)

$r_2 = 2-\sqrt{2}$, (125)

$r_1 = 1-\sqrt{(2\sqrt{2}-1)}$. (126)

Differentiating, we find

$$\frac{dI}{dr} = \tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} - \tfrac{1}{2}\frac{d\theta}{dr}, \quad (127)$$

so that

$$\theta = \tfrac{1}{3}\sqrt{2}\int_{r_0}^{r}\frac{dr}{\sqrt{R}} - 2I$$

$$= \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,F\phi - 2I. \quad (128)$$

When $r = \infty$, $F\phi = \tfrac{1}{2}K$,

and

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)} = \frac{\pi}{24}, \quad (129)$$

so that Θ, the corresponding value of θ, is given by

$$\Theta = \tfrac{1}{6}\sqrt{2}\sqrt{(2-\kappa^2)}\,\mathrm{K} - \frac{\pi}{12}$$

$$= \frac{\pi}{12}\left\{\sqrt{2}\sqrt{(2-\kappa^2)}\,\frac{\mathrm{K}}{\frac{1}{2}\pi} - 1\right\}. \quad (130)$$

In the branch of the catenary extending from $-\infty$ to r_1, in which r is negative, we must write

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r_0-r . r_1-r . r_3-r)}}{(-r)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r_3+r)}\sqrt{(r_2-r)}}{(-r)^{\frac{3}{2}}}. \quad (131)$$

In the closed branch of the catenary, we take

$$I = \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r_0-r.r_3-r.r-r_1)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{(r+r_2)\sqrt{(r-r_2)}}{r^{\frac{3}{2}}}, \qquad (132)$$

leading to

$$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \qquad (133)$$

and then

$$\theta - 2I = \tfrac{1}{3}\sqrt{2}\int_{r_2}\frac{dr}{\sqrt{R}} = \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,F\phi$$

$$= \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_2}{r_3-r_2.r-r_1}\right)},\ \&c., \qquad (134)$$

and, as r ranges from r_2 to r_3, the apsidal angle is given by

$$\Theta = \frac{\pi}{3}\left\{\tfrac{1}{2}\sqrt{2}\sqrt{(2-\kappa^2)}\,\frac{\mathrm{K}}{\frac{1}{2}\pi} + 1\right\}. \qquad (135)$$

With the parameters

$$v_1 = \omega_1 + \tfrac{1}{4}\omega_3,\quad v = \omega_1 + \tfrac{1}{2}\omega_3, \qquad (136)$$

we have already shown that

$$e = \sqrt{5}-2 = \left(\frac{\sqrt{5}-1}{2}\right)^3, \qquad (137)$$

$$b^4 = 4\sqrt{5}-8, \qquad (138)$$

and now

$$\kappa = \tfrac{1}{2}(\sqrt{5}-1) = \sin 38^\circ 10'. \qquad (139)$$

Now $\kappa = \kappa'^2$, $e = \kappa'^6$; and we find that

$$r_0 = 1+\kappa'+\kappa'^2, \qquad (140)$$

$$r_3 = 1+\kappa'-\kappa'^2, \qquad (141)$$

$$r_2 = 1-\kappa'+\kappa'^2, \qquad (142)$$

$$r_1 = 1-\kappa'-\kappa'^2; \qquad (143)$$

and it is also found that

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r+1-\kappa'^3)\sqrt{(r-r_0.r-r_3)}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1+\kappa'^3)\sqrt{(r-r_1.r-r_2)}}{r^2\sqrt{2}}, \qquad (144)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\kappa'(1+\kappa')\,\frac{1}{\sqrt{R}} - \frac{\kappa'^3}{r\sqrt{R}}; \qquad (145)$$

so that, in the infinite branch from r_0 to ∞,

$$\theta+2I=\tfrac{1}{2}\kappa'(1+\kappa'^2)\int_{r_0}\frac{dr}{\sqrt{R}}=\frac{F\phi}{2\kappa}$$

$$=\frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_0}{r_0-r_1.r-r_3}\right)},\ \&c. \qquad (146)$$

We can also write the relations

$$\left.\begin{aligned} r^2\sin\left(\frac{F\phi}{\kappa}-2\theta\right)&=\frac{1}{\sqrt{2}}(r+1-\kappa'^3)\sqrt{\{r^2-2(1+\kappa')r+2\kappa'(1+\kappa')\}}\\ r^2\cos\left(\frac{F\phi}{\kappa}-2\theta\right)&=\frac{1}{\sqrt{2}}(r+1+\kappa'^3)\sqrt{\{r^2-2(1-\kappa')r-2\kappa'(1-\kappa')\}}\end{aligned}\right\}. \qquad (147)$$

In passing from r_0 to ∞,

$$\Theta=\frac{\mathrm{K}}{4\kappa}-\frac{\pi}{8}=\frac{\pi}{8}\left(\frac{1}{\kappa}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}-1\right)=\frac{\pi}{8}\times 0{\cdot}8185. \qquad (148)$$

In the infinite branch from $-\infty$ to r_1, it is preferable to change the sign of the r's, and to write

$$I=\tfrac{1}{4}\sin^{-1}\frac{(r-1-\kappa'^3)\sqrt{\{r^2+2(1-\kappa')r-2\kappa'(1-\kappa')\}}}{r^2\sqrt{2}}$$

$$=\tfrac{1}{4}\cos^{-1}\frac{(r-1+\kappa'^3)\sqrt{\{r^2+2(1+\kappa')r+2\kappa'(1+\kappa')\}}}{r^2\sqrt{2}}, \qquad (149)$$

leading, on differentiation, to

$$\frac{dI}{dr}=\tfrac{1}{4}\kappa'(1+\kappa')\frac{1}{\sqrt{R}}+\frac{\kappa'^3}{r\sqrt{R}}, \qquad (150)$$

so that $$2I-\theta=\frac{F\phi}{2\kappa}$$

$$=\frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_2-r_0.r-r_1}{r_1-r_0.r-r_2}\right)},\ \&c., \qquad (151)$$

and, as r increases from r_1 to ∞, $4I$ increases from 0 to $\frac{5}{4}\pi$, and $F\phi$ from 0 to $\frac{1}{2}\mathrm{K}$, so that

$$\Theta=\tfrac{5}{8}\pi-\frac{\mathrm{K}}{4\kappa}$$

$$=\frac{\pi}{8}\left(5-\frac{1}{\kappa}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}\right)=\frac{\pi}{8}\times 3{\cdot}1815. \qquad (152)$$

In the limited branch, take

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r+1+\kappa'^2)\sqrt{(r-r_2.r-r_1)}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1-\kappa'^2)\sqrt{(r_0-r.r_3-r)}}{r^2\sqrt{2}}, \qquad (153)$$

and then $$\frac{dI}{dr} = -\tfrac{1}{4}\kappa'(1+\kappa'^2)\frac{1}{\sqrt{R}} + \frac{\kappa'^3}{r\sqrt{R}}, \qquad (154)$$

so that $$\theta - 2I = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_2}{r_3-r_2.r-r_1}\right)}, \text{ &c.,} \qquad (155)$$

or, as it may be written

$$\left.\begin{aligned} r^2\sin\left(2\theta-\frac{F\phi}{\kappa}\right) &= \frac{1}{\sqrt{2}}(r+1+\kappa'^2)\sqrt{(r-r_2.r-r_1)} \\ r^2\cos\left(2\theta-\frac{F\phi}{\kappa}\right) &= \frac{1}{\sqrt{2}}(r+1-\kappa'^2)\sqrt{(r_0-r.r_3-r)} \end{aligned}\right\}. \qquad (156)$$

As r grows from r_2 to r_3, the apsidal angle

$$\Theta = \frac{\pi}{4}\left(\frac{1}{\kappa}\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right) = \frac{\pi}{4}\times 2{\cdot}8185. \qquad (157)$$

With $$v_1 = \omega_1 + \tfrac{1}{5}\omega_3, \quad v = \omega_1 + \tfrac{2}{5}\omega_3, \qquad (158)$$

$$\wp 5v = e_1, \qquad (159)$$

so that, from (330), p. 450, *L.M.S.*, XXVII,

$$(1-2m)\,a = m^2(1-m), \qquad (160)$$

and making use of the values of a and m in (73) and (74), § 29, and putting $e^2 = c$, there results the sextic equation

$$c^6 + 194c^5 - 745c^4 + 6908c^3 + 2015c^2 - 4318c + 41 = 0, \qquad (161)$$

requiring solution.

The corresponding form of $I(v_1)$ must be

$$I = \tfrac{1}{5}\sin^{-1}\frac{(Hr+H_1)\sqrt{(r-r_0.r-r_1.r-r_3)}}{r^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\cos^{-1}\frac{(Kr^2+K_1r+K_2)\sqrt{(r-r_2)}}{r^{\frac{5}{2}}}, \qquad (162)$$

in which the determination of the H's and K's will give some trouble.

If we try $$v_1 = \omega_1 + \tfrac{1}{6}\omega_3, \quad v = \omega_1 + \tfrac{1}{3}\omega_3, \tag{163}$$

then $$\wp 3v = e_2; \tag{164}$$

and, from (326), p. 450, *L.M.S.*, XXVII,

$$s_1 - s_2 = \frac{(1-m)^2\, a}{a-m}, \tag{165}$$

leading to the equation

$$(e-1)(e^4 - 8e^3 - 18e^2 - 24e + 1)^2 = 0. \tag{166}$$

The corresponding $I(v_1)$ must be of the form

$$I = \tfrac{1}{6}\sin^{-1}\frac{(Hr^2 + H_1 r + H_2)\sqrt{(r-r_0 . r-r_3)}}{r^3}$$

$$= \tfrac{1}{6}\cos^{-1}\frac{(Kr^3 + K_1 r + K_2)\sqrt{(r-r_1 . r-r_2)}}{r^3}. \tag{167}$$

The parameters $$v_1 = \omega_1 + \tfrac{1}{7}\omega_3, \quad v = \omega_1 + \tfrac{2}{7}\omega_3 \tag{168}$$

make $$s(7v) = s_1, \tag{169}$$

so that, from (332), p. 450, *L.M.S.*, XXVII,

$$(1-2m)^3 a^2 - m(1-m)(1-2m)(2-3m)\, a + m^3 (1-m)^4 = 0, \tag{170}$$

giving an equation for e^2; and now

$$I = \tfrac{1}{7}\sin^{-1}\frac{(Hr^2 + H_1 r + H_2)\sqrt{(r-r_0 . r-r_1 . r-r_3)}}{r^{\frac{7}{2}}}$$

$$= \tfrac{1}{7}\cos^{-1}\frac{(Kr^3 + K_1 r^2 + K_2 r + K_3)\sqrt{(r-r_2)}}{r^{\frac{7}{2}}}, \tag{171}$$

and so on, for higher values; but the complexity, as is seen, increases very rapidly.

31. If $b > 1$, two roots of R are imaginary; we must now replace e^2 by $-e^2$, and e by ie in the preceding, and put

$$R = r^3 (r-2)^2 - 1 - c^2$$

$$= r^3 (r-2)^2 - b^4, \tag{172}$$

and $$r_0 = 1 + \sqrt{(b^2+1)}, \tag{173}$$

$$r_3 = 1 + i\sqrt{(b^2-1)}, \tag{174}$$

$$r_2 = 1 - i\sqrt{(b^2-1)}, \tag{175}$$

$$r_1 = 1 - \sqrt{(b^2+1)}. \tag{176}$$

Also $$e_1 = \tfrac{1}{3}, \quad e_2, e_3 = -\tfrac{1}{6} \pm \tfrac{1}{2}ie, \tag{177}$$

$$\kappa^2, \kappa'^2 = \tfrac{1}{2}\left(1 \mp \frac{1}{b^2}\right) = \tfrac{1}{2}\left\{1 \mp \frac{1}{\sqrt{(1+e^2)}}\right\}, \tag{178}$$

and $$u = \int_{r_0}^{r} \frac{dr}{\sqrt{R}} = \frac{F\phi}{b\sqrt{2}} = \sqrt{(\tfrac{1}{2} - \kappa^2)}\, F\phi, \tag{179}$$

where $$\tan^2 \tfrac{1}{2}\phi = \frac{r - r_0}{r - r_1}, \tag{180}$$

$$\cos\phi = \frac{r_0 - r_1}{2r - r_0 - r_1}. \tag{181}$$

These catenaries are not so interesting to draw, as the limited branch is absent; but a few specimens are added here.

Thus, for instance, for a parameter

$$v_1 = \tfrac{2}{3}\omega_2', \tag{182}$$

we have, changing the sign of $\sqrt{2}$ in (116), § 30,

$$b^2 = 2\sqrt{2} + 2, \tag{183}$$

$$\kappa^2 = \tfrac{1}{4}(3 - \sqrt{2}), \quad \kappa'^2 = \tfrac{1}{4}(\sqrt{2} + 1), \tag{184}$$

$$r_0 = 2 + \sqrt{2}, \qquad r_1 = -\sqrt{2}, \tag{185}$$

and we must take

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\, \frac{(r + 2 + \sqrt{2})\sqrt{(r - 2 - \sqrt{2})}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\, \frac{\sqrt{(r + \sqrt{2}.r^2 - 2r + 2\sqrt{2} + 2)}}{r^{\frac{3}{2}}}; \tag{186}$$

for, differentiating, $$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{187}$$

so that $$\theta - 2I = \tfrac{1}{3}\sqrt{2}\int_{r_0}^{r} \frac{dr}{\sqrt{R}} = \tfrac{1}{3}\sqrt{2}\sqrt{(\tfrac{1}{2} - \kappa^2)}\, F\phi. \tag{188}$$

With a parameter $$v_1 = \tfrac{1}{4}\omega_2', \tag{189}$$

we shall find that $$b^2 = 2, \tag{190}$$

$$\kappa^2 = \tfrac{1}{4}, \quad \kappa = \tfrac{1}{2} = \sin 30^\circ; \tag{191}$$

and $$I = \tfrac{1}{4}\sin^{-1}\frac{\sqrt{(r^2 - 2r - 2)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1)\sqrt{(r^2 - 2r + 2)}}{r^2}, \tag{192}$$

and $$\frac{dI}{dr} = -\tfrac{1}{4}\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{193}$$

so that $$\theta - 2I = \tfrac{1}{2}\int_{r_0}\frac{dr}{\sqrt{R}} = \tfrac{1}{4}F\varphi\,; \tag{194}$$

and this can be replaced by

$$r^2 \sin(2\theta - \tfrac{1}{2}F\phi) = \sqrt{(r^2-2r-2)}, \tag{195}$$

$$r^2 \cos(2\theta - \tfrac{1}{2}F\phi) = (r+1)\sqrt{(r^2-2r+2)}, \tag{196}$$

giving the development of a catenary on a cone, consisting of two sets of separate infinite branches.

Again, with $$v = \tfrac{2}{3}\omega_2',$$

$$I = \tfrac{1}{3}\sin^{-1}\frac{\left\{r^2-(\sqrt{3}-1)\,r-\frac{2}{\sqrt{3}}\right\}\sqrt{\left(r^2-2r-\frac{2}{\sqrt{3}}\right)}}{r^3\sqrt{2}}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{\left\{r^2+(\sqrt{3}+1)\,r+\frac{2}{\sqrt{3}}\right\}\sqrt{\left(r^2-2r+\frac{2}{\sqrt{3}}\right)}}{r^3\sqrt{2}}, \tag{197}$$

$$\frac{dI}{dr} = \frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{R}} - \frac{2}{\sqrt{3}}\,\frac{1}{r\sqrt{R}}, \tag{198}$$

so that, taking $$b^2 = \frac{2}{\sqrt{3}}, \tag{199}$$

we can make $$\theta + I = \frac{1}{\sqrt{3}}\int_{r_0}\frac{dr}{\sqrt{R}} = \frac{F\phi}{2\sqrt[4]{3}}, \tag{200}$$

and $$\kappa^2 = \tfrac{1}{2}(1-\tfrac{1}{2}\sqrt{3}), \quad \kappa = \sin 15°. \tag{201}$$

Whirling Catenary on a Cone.

32. If the cone is made to rotate about its axis with sufficient angular velocity to make gravity insensible compared with the centrifugal force, the differential equation of the catenary becomes changed into

$$\frac{d\theta}{dr^2} = \tfrac{1}{2}\frac{b^3}{r^2\sqrt{R}}, \tag{1}$$

where $$R = 4r^2(r^2-a^2)^2 - b^6, \tag{2}$$

giving the relation between θ and r in the plane development of the catenary.

Now
$$r\frac{d\theta}{dr} = \tan\phi = \frac{b^3}{\sqrt{R}}, \tag{3}$$

$$\sin\phi = \frac{b^3}{2r\,(r^2-a^2)}, \tag{4}$$

so that
$$p\,(r^2-a^2) = \tfrac{1}{2}b^3 \tag{5}$$

in the plane development.

This gives, as a particular case, when the cone itself is a plane, the catenary formed on a whirling horizontal table, or the catenary under a central force varying as the distance.

To reduce the relation (1) to the standard form of (A), put

$$y+1 = 0, \tag{6}$$

$$r^2 = M^2 s, \tag{7}$$

$$r^2-a^2 = M^2\,(s+x), \tag{8}$$

$$b^6 = -M^6\Sigma = M^6 x^3, \tag{9}$$

&c.,

and the analytical development is practically the same as that which follows for the trajectory of a particle on a smooth vertical cone.

The Trajectory of a Particle on a Cone.

33. The path of a particle on a smooth vertical cone (which can be imitated by rolling a coin inside a conical cardboard lamp-shade, or by the path of a bicycle on the banked conical turnings of a racing track) is given by

$$x^2\frac{d\psi}{dt} = H, \tag{1}$$

$$\sec^2\alpha\frac{dx^2}{dt^2} + x^2\frac{d\psi^2}{dt^2} = 2g\,(z\sim h)$$
$$= 2g\cot\alpha\,(x\sim a), \tag{2}$$

$$\sec^2\alpha\frac{dx^2}{dt^2} = 2g\cot\alpha\,(x\sim a) - \frac{H^2}{x^2}, \tag{3}$$

so that
$$\sin^2\alpha\frac{x^2 d\psi}{dx^2} = \frac{H^2}{2g\cot\alpha\,(x\sim a)\,x^2 - H^2}$$
$$= \frac{k^3}{4\,(x\sim a)\,x^2 - k^3}, \tag{4}$$

on putting
$$H^2 = \tfrac{1}{2}gk^3\cot\alpha\,;$$

or
$$\frac{r\,d\theta}{dr} = \frac{b^{\frac{3}{2}}}{\sqrt{\{4(r\sim a)\,r^2 - b^3\}}} = \frac{b^{\frac{3}{2}}}{\sqrt{R}} \tag{5}$$

in the developed curve, where

$$R = 4(r\sim a)\,r^2 - b^3. \tag{6}$$

Then
$$\sin\varphi = \frac{b^{\frac{3}{2}}}{2r\sqrt{(r\sim a)}}, \tag{7}$$

so that
$$p = r\sin\varphi = \frac{b^{\frac{3}{2}}}{\sqrt{(r\sim a)}}, \tag{8}$$

or
$$p^2(r\sim a) = b^3, \tag{9}$$

is the curve which is the development of the path of the particle on the cone.

Since p has no stationary values, this curve has no points of inflexion; but the projection on a horizontal plane of the trajectory on the cone can have points of inflexion.

In particular, if $a = 0$, the curve becomes

$$2r^{\frac{3}{2}}\cos\tfrac{3}{2}\theta = b^{\frac{3}{2}}, \tag{10}$$

a central orbit described under a constant central force.

The arc s is given by

$$s = \int \frac{r\sqrt{(r\sim a)}\,dr}{\sqrt{\{4(r\sim a)\,r - b^3\}}}, \tag{11}$$

an integral similar to those occurring with the paraboloid.

If the developed curve is described as a central orbit, with velocity v,

$$v^2 = \frac{h^2}{p^2} = \frac{h^2}{b^3}(r\sim a), \tag{12}$$

and the central force, given by $d\frac{1}{2}v^2/dr$, is constant.

So also, if the developed catenary on the cone is treated as a plane catenary under a central force at the origin, and if T denotes the tension

$$T = \frac{H}{p} = \frac{H}{b^2}(r - 2a), \tag{13}$$

and the central force, given by dT/dr, is also constant.

But, if this catenary is described as a central orbit, the force varies as $r - 2a$; and, if $a = 0$, we obtain the rectangular hyperbola, as before.

With a central force (realisable with an elastic thread)

$$P = ar+b, \tag{14}$$

$$v^2 = ar^2+2br+c, \tag{15}$$

$$= \frac{h^2}{p^2} = h^2\left(\frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{d\theta^2}\right), \tag{16}$$

so that

$$\frac{h^2}{r^2}\frac{dr^2}{d\theta^2} = ar^4+2br^3+cr^2+0-h^2. \tag{17}$$

Or, more generally, with (Legendre, *F.E.I.*, p. 557)

$$P = ar+b+dr^{-2}, \tag{18}$$

$$\frac{h^2}{r^2}\frac{dr^2}{d\theta^2} = ar^4+2br^3+cr^2-dr-h^2 = R, \tag{19}$$

suppose; or, with

$$r = 1/u,$$

$$h^2u^2\frac{du^2}{d\theta^2} = -h^2u^4-du^3+cu^2+2bu+a = U, \tag{20}$$

suppose; so that

$$\theta = \int \frac{hu\,du}{\sqrt{U}}, \tag{21}$$

a form which can be compared immediately with Abel's results.

Generally, if the trajectory on the cone is described under a central force P, acting through the vertex of the cone, we find that

$$\int P\,dr = \frac{h^2}{p^2}, \tag{22}$$

where

$$H^2 = h^2\sin^2\alpha, \tag{23}$$

so that P is the same as for the plane development of the trajectory, described as a central orbit, and this explains why the developed curve is described as a central orbit under a constant central force.

Again, for instance, the equation of the development of a plane section of the eight circular cone being of the form

$$\frac{l}{r} = 1+e\cos k\theta, \tag{24}$$

the central force

$$P = Ar^{-2}+Br^{-3}. \tag{25}$$

34. Comparing equation (5) with the standard form in (A), we must take

$$y+1=0; \tag{26}$$

or, employing the forms given on p. 449, *L.M.S.*, XXVII,

$$y+1=(2a-1)\frac{m}{a-m}, \tag{27}$$

so that either $m=0$, which must be rejected, or

$$a=\tfrac{1}{2}; \tag{28}$$

and this makes

$$x=-\frac{2m^3}{(1-2m)^2}, \tag{29}$$

$$s_\gamma=\frac{m^2}{(1-2m)^2}, \tag{30}$$

$$(s_\alpha-s_\gamma)(s_\beta-s_\gamma)=\frac{m^4(3-2m)}{(1-2m)^3}, \tag{31}$$

$$(s_\alpha-s_\beta)^2=\frac{m^4(8m-3)}{(1-2m)^4}, \tag{32}$$

$$(s_\alpha-s_\gamma+s_\beta-s_\gamma)^2=\frac{m^4(4m-3)^2}{(1-2m)^4}, \tag{33}$$

$$s_\alpha-s_\gamma,\ s_\beta-s_\gamma=\frac{m^2}{2(1-2m)^2}\{4m-3\pm\sqrt{(8m-3)}\}, \tag{34}$$

$$s_\alpha,\ s_\beta=\frac{m^2}{2(1-2m)^2}\{4m-1\pm\sqrt{(8m-3)}\}. \tag{35}$$

When $8m-3$ is positive, the parameter

$$v=\omega_1+f\omega_3, \tag{36}$$

and there are three cases to distinguish :

(i.) When $1/f$ is an odd integer,

$$s_\alpha>s_\gamma>s_\beta, \tag{37}$$

$$\kappa^2=\frac{s_\gamma-s_\beta}{s_\alpha-s_\beta}=\frac{3-4m+\sqrt{(8m-3)}}{2\sqrt{(8m-3)}}, \tag{38}$$

$$\kappa'^2=\frac{s_\alpha-s_\gamma}{s_\alpha-s_\beta}=-\frac{3+4m+\sqrt{(8m-3)}}{2\sqrt{(8m-3)}}, \tag{39}$$

$$s_\gamma-s(v)=\frac{m^2a}{a-m}=\frac{m^2}{1-2m}, \tag{40}$$

and this must be negative; so that

$$m>\tfrac{1}{2}. \tag{41}$$

(ii.) When $1/f$ is an even integer,

$$s_\alpha > s_\beta > s_\gamma, \tag{42}$$

$$\kappa^2 = \frac{s_\beta - s_\gamma}{s_\alpha - s_\gamma} = \frac{4m-3-\sqrt{(8m-3)}}{4m-3+\sqrt{(8m-3)}}, \tag{43}$$

$$\kappa'^2 = \frac{s_\alpha - s_\beta}{s_\alpha - s_\gamma} = \frac{2\sqrt{(8m-3)}}{4m-3+\sqrt{(8m-3)}}, \tag{44}$$

and $s_\gamma - s(v)$ is negative; so that

$$m > \tfrac{3}{4}. \tag{45}$$

(iii.) When $1/f$ is half an odd integer,

$$s_\gamma > s_\alpha > s_\beta,$$

$$\kappa^2 = \frac{s_\alpha - s_\beta}{s_\gamma - s_\beta} = \frac{2\sqrt{(8m-3)}}{-4m+3+\sqrt{(8m-3)}}, \tag{46}$$

$$\kappa'^2 = \frac{s_\gamma - s_\alpha}{s_\gamma - s_\beta} = \frac{-4m+3-\sqrt{(8m-3)}}{-4m+3+\sqrt{(8m-3)}}, \tag{47}$$

and $s_\gamma - s(v)$ is positive, so that

$$\tfrac{1}{2} > m > \tfrac{3}{8}. \tag{48}$$

When $8m-3$ is negative, the parameter

$$v = f\omega_2', \tag{49}$$

and

$$\kappa^2,\ \kappa'^2 = \tfrac{1}{2}\left\{1 \mp \frac{s_\gamma - \frac{1}{2}(s_\alpha + s_\beta)}{\sqrt{(s_\alpha - s_\gamma . s_\beta - s_\gamma)}}\right\} \tag{50}$$

$$= \tfrac{1}{2}\left\{1 \mp \frac{3-4m}{2\sqrt{(3-2m . 1-2m)}}\right\}. \tag{51}$$

In the upper limited branch of the trajectory, described by the particle sliding on the inside of the upper sheet of the cone, take

$$R = 4(a-r)r^2 - b^3$$

$$= 4(r_1 - r . r_2 - r . r_3 - r), \tag{52}$$

$$\infty > r_3 > r > r_2 > 0 > r_1 > -\infty. \tag{53}$$

Writing σ for $s(v)$, then, we put

$$r = -M^2(s+x) = M^2(\sigma - s), \tag{54}$$

$$a - r = M^2 s, \tag{55}$$

$$R = M^6 S,$$

$$a = -M^2 x, \tag{56}$$

$$b^3 = -M^6\Sigma = M^6x^3, \tag{57}$$

$$x = -\frac{a^3}{b^3}, \tag{58}$$

$$M^2 = \frac{b^3}{a^2}, \tag{59}$$

$$M\frac{dr}{\sqrt{R}} = -\frac{ds}{\sqrt{S}}, \tag{60}$$

and now the standard integral (A)

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int_s^{s_2} \frac{P(\sigma - s) - \sqrt{(-\Sigma)}}{\sigma - s}\,\frac{ds}{\sqrt{S}} \\ &= \tfrac{1}{2}PM\int_{r_2}^{r}\frac{dr}{\sqrt{R}} - \tfrac{1}{2}\int\frac{b^{\frac{3}{2}}dr}{r\sqrt{R}} \\ &= \tfrac{1}{2}PM\,\frac{F\phi}{\sqrt{(r_3 - r_1)}} - \tfrac{1}{2}\theta, \end{aligned} \tag{61}$$

and

$$F\phi = \mathrm{dn}^{-1}\sqrt{\left(\frac{r_2 - r_1}{r - r_1}\right)}, \tag{62}$$

$$\kappa^2 = \frac{r_3 - r_2}{r_3 - r_1}, \tag{63}$$

so that

$$\Delta^2\phi = \frac{r_2 - r_1}{r - r_1}, \tag{64}$$

$$\sin^2\phi = \frac{r_3 - r_1}{r_3 - r_2}\cdot\frac{r - r_2}{r - r_1}. \tag{65}$$

35. Also, for the time integral,

$$r^2\frac{dr^2}{dt^2} = \tfrac{1}{2}g\cos a\, R, \tag{66}$$

or, putting

$$\tfrac{1}{2}g\cos a = ax^2,$$

$$\begin{aligned} nt &= \int_{r_2}^{r}\frac{r\,dr}{\sqrt{(aR)}} \\ &= \frac{1}{\sqrt{(a.r_3 - r_1)}}\int_0^{\phi}\left(r_1 + \frac{r_2 - r_1}{\Delta^2\phi}\right)\frac{d\phi}{\Delta\phi}, \end{aligned}$$

and (Legendre, *Fonctions Elliptiques*, I, p. 256)

$$\int_0^{\phi}\frac{d\phi}{\Delta^3\phi} = \frac{E\phi}{\kappa'^2} - \frac{\kappa^2}{\kappa'^2}\,\frac{\sin\phi\,\cos\phi}{\Delta\phi}. \tag{67}$$

The expression for the time is simpler if reckoned from the upper limit of the trajectory; and now

$$I(v) = \tfrac{1}{2}\int_s \frac{P(\sigma - s) - \sqrt{(-\Sigma)}}{\sigma - s}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_r^{r_3}\frac{dr}{\sqrt{R}} - \tfrac{1}{2}\int_r^{r_3}\frac{b^2\,dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\,\frac{F\phi}{\sqrt{(r_3 - r_1)}} - \tfrac{1}{2}\theta, \qquad (68)$$

where $F\phi = \mathrm{sn}^{-1}\sqrt{\left(\dfrac{r_3 - r}{r_3 - r_2}\right)} = \mathrm{cn}^{-1}\sqrt{\left(\dfrac{r - r_2}{r_3 - r_2}\right)} = \mathrm{dn}^{-1}\sqrt{\left(\dfrac{r - r_1}{r_3 - r_1}\right)}$; (69)

and

$$nt = \int_r^{r_3}\frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a.r_3 - r_1)}}\int_0^{\phi}\left\{r_1 + (r_3 - r_1)\,\Delta^2\phi\right\}\frac{d\phi}{\Delta\phi}$$

$$= \frac{1}{\sqrt{(a.r_3 - r_1)}}\left\{r_1 F\phi + (r_3 - r_1)\,E\phi\right\}. \qquad (70)$$

If $r_2 = r_3$, as in steady motion in a horizontal circle, then

$$r_1 = -\tfrac{1}{2}r_3,$$

and

$$\kappa = 0, \quad F\phi = E\phi = \phi;$$

so that

$$nt = \frac{r_3\phi}{\sqrt{(a.r_3 - r_1)}}, \qquad (71)$$

and the small oscillations synchronize with a simple pendulum of length l, such that

$$n\sqrt{\frac{l}{g}} = \frac{r_3}{\sqrt{\ a.r_3 - r_1)}},$$

or

$$l = \frac{g}{n^2}\,\frac{r_3^2}{a\,(r_3 - r_1)} = 2\sec a\,\frac{r_3^2}{r_3 - r_1} = \tfrac{4}{3}OG, \qquad (72)$$

if the normal to the cone meets the axis in G.

But, if the particle describes the same circle, suspended from G, as a conical or spherical pendulum, then it will be found that the small oscillations synchronize with a pendulum of length

$$\frac{OG}{1 + \frac{1}{4}\cot^2 a}. \qquad (73)$$

36. In the unlimited branch of the trajectory, described by the particle sliding on the outside of the lower sheet of the cone, it is convenient to change the sign of r, r_1, r_2, r_3, and to make

$$R = 4(r+a)r^3 - b^3$$

$$= 4.r-r_1.r-r_2.r-r_3, \tag{74}$$

$$\infty > r > r_1 > 0 > r_2 > r_3 > -\infty. \tag{75}$$

We now put $$r = M^2(s+x) = M^2(s-\sigma), \tag{76}$$

$$r+a = M^2 s, \tag{77}$$

$$R = M^6 S, \tag{78}$$

$$M\frac{dr}{\sqrt{R}} = \frac{ds}{\sqrt{S}}, \tag{79}$$

and now

$$I(v) = \tfrac{1}{2}\int_{s_1}^{s} \frac{P(s-\sigma)+\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_{r_1}^{r}\frac{dr}{\sqrt{R}} + \tfrac{1}{2}\int_{r_1}^{r}\frac{b^3\,dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\,\frac{F\phi}{\sqrt{(r_1-r_3)}} + \tfrac{1}{2}\theta, \tag{80}$$

where

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_2}\right)} = \mathrm{cn}^{-1}\sqrt{\left(\frac{r_1-r_2}{r-r_2}\right)} = \mathrm{dn}^{-1}\sqrt{\left(\frac{r_1-r_2}{r_1-r_3}\,\frac{r-r_3}{r-r_2}\right)}, \tag{81}$$

and, in the expression for the time,

$$nt = \int_{r_1}^{r}\frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a.r_1-r_3)}}\int_0^{\phi}\{r_2+(r_1-r_2)\sec^2\phi\}\,\frac{d\phi}{\Delta\phi}, \tag{82}$$

in which (Legendre)

$$\int_0 \sec^2\phi\,\frac{d\phi}{\Delta\phi} = \tan\phi\,\Delta\phi + \kappa^2 F\phi - E\phi. \tag{83}$$

We might also have started from $r=\infty$, and then

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r_1-r_3}{r-r_3}\right)} = \mathrm{cn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_3}\right)} = \mathrm{dn}^{-1}\sqrt{\left(\frac{r-r_2}{r-r_3}\right)}, \tag{84}$$

but now the expression for the time will introduce an infinite constant.

We notice that, while $r=0$ corresponds to the elliptic argument or parameter $u=v=\omega_1+f\omega_3$, the value of $r=a$ in the upper branch, or of $r=-a$ in the lower branch, corresponds to the argument

$$u=2v=2f\omega_3.$$

37. Suppose we begin by taking $f=\frac{1}{2}$; then $2v=\omega_3$, so that $r\mp a$ must be a factor of R, which requires $b=0$; and the trajectory lies in a vertical plane through the vertex.

Next take $f=\frac{1}{3}$, when $m=1$, (85)

$$x=-2,\quad y=-1,\quad M^2=\tfrac{1}{2}a,\quad b^2=\tfrac{1}{2}a^2, \tag{86}$$

[*L.M.S.*, XXVII, (364), p. 460]; and, from (38) and (39), § 34,

$$\kappa^2,\ \kappa'^2=\frac{\sqrt{5}\mp1}{2\sqrt{5}}. \tag{87}$$

In the upper limited branch of the trajectory,

$$r=\tfrac{1}{2}a\,(2-s), \tag{88}$$

$$a-r=\tfrac{1}{2}as, \tag{89}$$

$$\begin{aligned} R&=4\,(a-r)\,r^2-\tfrac{1}{2}a^3\\ &=(r-\tfrac{1}{2}a)(-4r^2+2ar+a^2), \end{aligned} \tag{90}$$

$$r_3=\tfrac{1}{4}\,(\sqrt{5}+1)\,a,\quad r_2=\tfrac{1}{2}a,\quad r_1=-\tfrac{1}{4}\,(\sqrt{5}-1)\,a, \tag{91}$$

and the corresponding pseudo-elliptic integral

$$\begin{aligned} I(\omega_1+\tfrac{1}{3}\omega_3)&=\tfrac{1}{3}\sin^{-1}\frac{(r+\frac{1}{2}a)\sqrt{(r-\frac{1}{2}a)}}{r^{\frac{3}{2}}}\\ &=\tfrac{1}{3}\cos^{-1}\frac{\sqrt{\{\frac{1}{2}a\,(-r^2+\frac{1}{2}ar+\frac{1}{4}a^2)\}}}{r^{\frac{3}{2}}}, \end{aligned} \tag{92}$$

$$\frac{dI}{dr}=-\tfrac{1}{6}\sqrt{2}\,\frac{\sqrt{a}}{\sqrt{R}}+\frac{1}{2\sqrt{2}}\,\frac{a^{\frac{3}{2}}}{r\sqrt{R}}, \tag{93}$$

so that

$$\begin{aligned} \theta&=\tfrac{1}{3}\sqrt{2}\int\frac{\sqrt{a}\,dr}{\sqrt{R}}+2I\\ &=\frac{2F\phi}{3\sqrt[4]{5}}+2I, \end{aligned} \tag{94}$$

$$F\phi=\mathrm{dn}^{-1}\sqrt{\left\{\frac{\frac{1}{4}\,(\sqrt{5}+1)\,a}{r+\frac{1}{4}\,(\sqrt{5}-1)\,a}\right\}},\ \text{&c.},$$

and the apsidal angle

$$\Theta=\frac{\pi}{3}\left(\frac{1}{\sqrt[4]{5}}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right). \tag{95}$$

Changing the sign of r in the open unlimited branch of the trajectory on the lower half of the cone,

$$r = \tfrac{1}{2}a(s-2), \tag{96}$$

$$r+a = \tfrac{1}{2}as, \tag{97}$$

$$R = 4(r+a)r^2 - \tfrac{1}{2}a^3$$

$$= (r+\tfrac{1}{2}a)(4r^2+2ar-a^2), \tag{98}$$

$$r_1 = \tfrac{1}{4}(\sqrt{5}-1)a, \quad r_2 = -\tfrac{1}{2}a, \quad r_3 = -\tfrac{1}{4}(\sqrt{5}+1)a, \tag{99}$$

and now the corresponding

$$I = \tfrac{1}{3}\sin^{-1}\frac{\sqrt{\{\tfrac{1}{2}a(r^2+\tfrac{1}{2}ar-\tfrac{1}{4}a^2)\}}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{(-r+\tfrac{1}{2}a)\sqrt{(r+\tfrac{1}{2}a)}}{r^{\frac{3}{2}}}, \tag{100}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{6}\sqrt{2}\frac{\sqrt{a}}{\sqrt{R}} + \frac{1}{2\sqrt{2}}\frac{a^{\frac{3}{2}}}{r\sqrt{R}}, \tag{101}$$

so that

$$\theta = -\frac{2F\phi}{3\sqrt[4]{5}} + 2I, \tag{102}$$

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_2}\right)}, \text{ &c.,} \tag{103}$$

and now the apsidal angle, in going from r_1 to infinity is given by

$$\Theta = -\frac{2\mathrm{K}}{3\sqrt[4]{5}} + \frac{2\pi}{3}$$

$$= \frac{\pi}{3}\left(2 - \frac{1}{\sqrt[4]{5}}\frac{\mathrm{K}}{\frac{1}{2}\pi}\right). \tag{104}$$

With $f = \frac{1}{4}$, and

$$v = \omega_1 + \tfrac{1}{4}\omega_5, \tag{105}$$

the equation obtained from putting in (329) (*L.M.S.*, XXVII, p. 450) $s_7 - s_4 = 0$ and $\alpha = \frac{1}{2}$ gives

$$2m^2 - 4m + 1 = 0 \tag{106}$$

or

$$m = 1 \pm \sqrt{2} \tag{107}$$

of which the upper sign must be taken so as to make $m > \frac{3}{4}$ in (45); and then

$$\frac{b^3}{a^3} = -\frac{1}{x} = 2 - \sqrt{2}. \tag{108}$$

Now, in the upper limited branch of the trajectory,

$$R = 4(a-r)r^2-(2-\sqrt{2})a^3$$
$$= 4\left(\frac{a}{\sqrt{2}}-r\right)\left(r^2-\frac{2-\sqrt{2}}{2}ar-\frac{\sqrt{2}-1}{2}a^2\right), \quad (109)$$

so that $$r_3 = \frac{a}{\sqrt{2}}, \quad r_2,\ r_1 = \frac{\sqrt{2}-1\pm\sqrt{(2\sqrt{2}-1)}}{2\sqrt{2}}a; \quad (110)$$

and we shall find that the associated pseudo-elliptic integral is

$$I(\omega_1+\tfrac{1}{4}\omega_3) = \tfrac{1}{4}\sin^{-1}\frac{\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\{r+(\sqrt{2}-1)a\}\sqrt{\left(\frac{a}{\sqrt{2}}-r\right)}}{r^2}$$
$$= \tfrac{1}{4}\cos^{-1}\frac{(r+a)\sqrt{\{r^2-\frac{1}{2}(2-\sqrt{2})ar-\frac{1}{2}(\sqrt{2}-1)a^2\}}}{r^2}, \quad (111)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\sqrt{a}}{\sqrt{R}} - \tfrac{1}{2}\frac{d\theta}{dr}, \quad (112)$$

so that $$\theta = \tfrac{1}{2}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\int_r^{r_3}\frac{\sqrt{a}\,dr}{\sqrt{R}} - 2I$$
$$= \tfrac{1}{2}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\sqrt{\left(\frac{a}{r_3-r_1}\right)}F\phi - 2I, \quad (113)$$

where $$\sin^2\phi = \frac{r_3-r}{r_3-r_2}, \text{ \&c.}, \quad (114)$$

and, on reduction,

$$\sqrt{\left(\frac{a}{r_3-r_1}\right)} = \frac{2\sqrt[4]{2}}{\sqrt{(\sqrt{2}+1)}+\sqrt{(\sqrt{2}-1)}\sqrt{(2\sqrt{2}-1)}}, \quad (115)$$

so that $$\theta = mF\phi - 2I, \quad (116)$$

where $$m = \tfrac{1}{4}\{1-(\sqrt{2}-1)\sqrt{(2\sqrt{2}-1)}\}, \quad (117)$$

and the apsidal angle $$\Theta = mK - \tfrac{1}{4}\pi. \quad (118)$$

In the lower unlimited branch, changing the sign of r, so that

$$R = 4(r+a)r^2-(2-\sqrt{2})a^3, \quad (119)$$

$$r_1,\ r_2 = \frac{\pm\sqrt{(2\sqrt{2}-1)}-\sqrt{2}+1}{2\sqrt{2}}a, \quad r_3 = -\frac{a}{\sqrt{2}}, \quad (120)$$

and we take

$$I = \tfrac{1}{4}\sin^{-1}\frac{(a-r)\sqrt{(r-r_1 . r-r_2)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\{r-(\sqrt{2}-1)\,a\}\sqrt{(r-r_2)}}{r^2}, \quad (121)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\sqrt{a}}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \quad (122)$$

so that
$$\theta = 2I - mF\phi, \quad (123)$$

where
$$\sin^2\phi = \frac{r-r_1}{r-r_2}, \text{ &c.}, \quad (124)$$

and now the apsidal angle, in passing from r_1 to infinity, is given by

$$\Theta = \tfrac{3}{4}\pi - m\mathrm{K}. \quad (125)$$

38. The other value of m,

$$m = 2 - \tfrac{1}{2}\sqrt{2}, \quad (126)$$

makes
$$\frac{b^3}{a^3} = 2+\sqrt{2}, \quad (127)$$

and R has now two imaginary roots, so that the trajectory consists of a single infinite branch.

As another specimen of a trajectory of this character, we may take the one based upon $I(\frac{2}{5}\omega_2')$, in which

$$x = y = -1, \quad (128)$$

so that
$$S = 4s\,(s-1)^2 - 1, \quad (129)$$

as in the Transformation of the Eleventh Order (*L.M.S.*, XXVII, p. 424), and in Abel's *Œuvres*, III, p. 142.

We have now to put

$$r = M^2(s-1), \quad (130)$$

$$r+a = M^2 s, \quad (131)$$

$$a = M^2, \quad (132)$$

$$b^3 = a^3, \text{ &c.}, \quad (133)$$

in the pseudo-elliptic integral

$$I(\tfrac{2}{5}\omega_2') = \tfrac{1}{5}\cos^{-1}\frac{2s^2-3s}{2\,(s-1)^{\frac{3}{2}}}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(s-2)\sqrt{\{4s\,(s-2)^2-1\}}}{2\,(s-1)^{\frac{3}{2}}},$$

so that $$I = \tfrac{1}{5}\cos^{-1}\frac{2r^2+ar-a^2}{2r^{\frac{1}{2}}}\sqrt{a}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(r-a)\sqrt{\{4(r+a)r^2-a^5\}}}{2r^{\frac{1}{2}}}, \tag{134}$$

$$= \tfrac{2}{5}\sin^{-1}\frac{(r^{\frac{1}{2}}-a^{\frac{1}{2}})\sqrt{(2r^{\frac{3}{2}}+2a^{\frac{1}{2}}r+2ar^{\frac{1}{2}}+a^{\frac{3}{2}})}}{2r^{\frac{1}{4}}}$$

$$= \tfrac{2}{5}\cos^{-1}\frac{(r^{\frac{1}{4}}+a^{\frac{1}{4}})\sqrt{(2r^{\frac{3}{2}}-2a^{\frac{1}{2}}r+2ar^{\frac{1}{2}}-a^{\frac{3}{2}})}}{2r^{\frac{1}{4}}}, \tag{135}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{5}\frac{\sqrt{a}}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{136}$$

so that $$\theta = 2I - \tfrac{1}{5}\int_{r_2}^{r}\frac{\sqrt{a}\,dr}{\sqrt{R}}. \tag{137}$$

To reduce this elliptic integral of the first kind to Legendre's function $F\phi$, we must determine the real root of $R=0$, or $S=0$; or else determine m from the cubic equation

$$2m^3-(2m-1)^2 = 0. \tag{138}$$

39. If the particle, instead of moving freely under gravity on the cone, is attached by a light thread to another particle, hanging freely at the end of the thread, the thread passing through a smooth hole at the vertex of the cone, there is no material change in the analytical equations; so that the preceding equations can be applied without essential modification. Thus, if the particles balance, the trajectory on the cone develops into a Cotes's spiral.

The cone, for instance, can be replaced by a flat smooth horizontal table, along which the particle is projected, attached to the second particle by a thread passing through a hole in the table.

When the particle is replaced by a ball, spinning and rolling on the cone, the dynamical equations are of the same form (Routh, *Rigid Dynamics*), but we have an additional constant at our disposal, as the restriction of (26), § 34, is no longer required.

If the cone is made to rotate with constant angular velocity about its axis, the function R rises from the third to the fourth degree, and the differential equations are of the same form as those given in § 33.

In a stereoscopic representation we should take a *thalweg* procession of particles spaced at equal time intervals (Hadamard, *Liouville*, 1897).

*The Geodesics on a Quadric of Revolution.**

40. These geodesics have been discussed recently by Prof. Forsyth, in the *Messenger of Mathematics*, 1896; they are introduced here as another illustration of the Elliptic Integral of the Third Kind in the notation employed in this paper.

In the geodesic on the oblate spheroid, generated by the revolution about Oy of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1}$$

$$x^2\frac{d\psi}{ds}=k, \tag{2}$$

or
$$ds^2=dx^2+dy^2+x^2d\psi^2=\frac{x^4}{k^2}d\psi^2; \tag{3}$$

and therefore, with
$$y=\frac{b}{a}\sqrt{(a^2-x^2)}, \tag{4}$$

$$dy^2=\frac{b^2}{a^2}\frac{x^2dx^2}{a^2-x^2}, \tag{5}$$

$$dx^2\left(1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}\right)+x^2d\psi^2\left(1-\frac{x^2}{k^2}\right)=0, \tag{6}$$

or
$$d\psi^2=\frac{k^2e^2\left(\frac{a^2}{e^2}-x^2\right)}{x^2(x^2-k^2)(a^2-x^2)}dx^2. \tag{7}$$

Put
$$x^2=n^2(s-\sigma), \tag{8}$$

$$\frac{a^2}{e^2}-x^2=m^2(s_1-s), \tag{9}$$

$$a^2-x^2=m^2(s_2-s), \tag{10}$$

$$x^2-k^2=m^2(s-s_3), \tag{11}$$

so that
$$a>x>k,$$

and
$$s_2>s>s_3>\sigma. \tag{12}$$

Since $s_2-\sigma$ is positive, the parameter v of the integral is a fraction of the imaginary period, or

$$v=f\omega_3. \tag{13}$$

Then
$$d\psi=\tfrac{1}{2}\frac{ke}{m}\frac{\sqrt{(s_1-s)}}{(s-\sigma)\sqrt{(s_2-s.s-s_3)}}ds, \tag{14}$$

* Legendre, *Fonctions Elliptiques*, t. I, § III, 360.

and, putting $x = 0, \quad s = \sigma,$

$$\frac{a^2}{e^2} = m^2(s_1 - \sigma), \tag{15}$$

$$a^2 = m^2(s_2 - \sigma), \tag{16}$$

$$k^2 = m^2(s_3 - \sigma), \tag{17}$$

$$e^2 = \frac{s_2 - \sigma}{s_3 - \sigma}, \tag{18}$$

$$\frac{k^2 e^2}{m^2} = \frac{s_2 - \sigma \,.\, s_3 - \sigma}{s_1 - \sigma}. \tag{19}$$

Then, with $$S = 4(s - s_1)(s - s_2)(s - s_3), \tag{20}$$

$$\Sigma = 4(\sigma - s_1)(\sigma - s_2)(\sigma - s_3), \tag{21}$$

$$\begin{aligned}
\frac{d\psi}{ds} &= \tfrac{1}{2}\sqrt{\left(\frac{s_2 - \sigma \,.\, s_3 - \sigma}{s_1 - \sigma}\right)} \frac{\sqrt{(s_1 - s)}}{(s - \sigma)\sqrt{(s_2 - s \,.\, s - s_3)}} \\
&= \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1 - \sigma} \frac{s_1 - s}{(s - \sigma)\sqrt{S}} \\
&= \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{(s - \sigma)\sqrt{S}} - \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\frac{1}{\sqrt{S}} \\
&= \tfrac{1}{2}\left(P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\right)\frac{1}{\sqrt{S}} - \tfrac{1}{2}\frac{P(s - \sigma) - \sqrt{(-\Sigma)}}{(s - \sigma)\sqrt{S}},
\end{aligned} \tag{22}$$

$$\psi = \tfrac{1}{2}\left(P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\right)u - I(v). \tag{23}$$

But P or $P(v)$ is connected with $P(\omega_1 + v)$ by the relation

$$P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma} = P(\omega_1 + v), \tag{24}$$

so that $$\psi = \tfrac{1}{2}\frac{P(\omega_1 + f\omega_3)}{\sqrt{(s_1 - s_3)}} F\phi - I(v), \tag{25}$$

where $$\sin^2\phi = \frac{s - s_3}{s_2 - s_3} = \frac{x^2 - k^2}{a^2 - k^2}, \quad \cos^2\phi = \frac{a^2 - x^2}{a^2 - k^2}, \quad \Delta^2\phi = \frac{a^2 - e^2x^2}{a^2 - e^2k^2}. \tag{26}$$

The first requirement of an algebraical geodesic is that

$$P(\omega_1 + f\omega_3) = 0; \tag{27}$$

but, as pointed out by Halphen (*F.E.*, II, p. 275), this relation implies a negative discriminant for the cubic S and two imaginary roots, which is excluded by the conditions of the problem; the same applies to the geodesics on all the other quadric surfaces.

We shall find that, in Glaisher's notation,

$$\frac{\frac{1}{2}P(\omega_1+f\omega_3)}{\sqrt{(s_1-s_3)}} = Z(f\mathrm{K}', \kappa') = \mathrm{zn}\, f\mathrm{K}', \quad \frac{\frac{1}{2}P(f\omega_3)}{\sqrt{(s_1-s_2)}} = \mathrm{zs}\, f\mathrm{K}', \text{ \&c.} \quad (28)$$

41. On the prolate spheroid, generated by the revolution of the ellipse about Ox, we find in a similar manner, from

$$y^2\frac{d\psi}{ds} = k, \quad (29)$$

$$d\psi = \tfrac{1}{2}\frac{kae}{b}\cdot\frac{y^2+\dfrac{b^4}{a^2e^2}}{y^2\sqrt{\left(b^2-y^2.y^2-k^2.y^2+\dfrac{b^4}{a^2e^2}\right)}}\,dy^2, \quad (30)$$

and we put

$$y^2 = m^2(\sigma - s), \quad (31)$$

$$b^2-y^2 = m^2(s-s_3), \quad (32)$$

$$y^2-k^2 = m^2(s_2-s), \quad (33)$$

$$y^2+\frac{b^4}{a^2e^2} = m^2(s_1-s). \quad (34)$$

Putting $y = 0$ or $s = \sigma$,

$$b^2 = m^2(\sigma - s_3), \quad (35)$$

$$k^2 = m^2(\sigma - s_2), \quad (36)$$

$$\frac{b^4}{a^2e^2} = m^2(s_1-\sigma), \quad (37)$$

$$\frac{k^2a^2e^2}{m^2b^2} = \frac{\sigma-s_2\,.\,\sigma-s_3}{s_1-\sigma}, \quad (38)$$

and

$$s_1 > \sigma > s_2 > s > s_3, \quad (39)$$

so that

$$v = \omega_1+f\omega_3, \quad u = \omega_3+f'\omega_1. \quad (40)$$

We now find

$$\frac{d\psi}{ds} = \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\frac{s_1-s}{(\sigma-s)\sqrt{S}}$$

$$= \tfrac{1}{2}\left(\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}+P\right)\frac{1}{\sqrt{S}} - \tfrac{1}{2}\frac{P(\sigma-s)-\sqrt{(-\Sigma)}}{(\sigma-s)\sqrt{S}}, \quad (41)$$

$$\psi = \tfrac{1}{2}\frac{P(f\omega_3)}{\sqrt{(s_1-s_3)}}F\phi - I(v), \quad (42)$$

with $\sin^2\phi = \dfrac{b^2-y^2}{b^2-k^2}$, $\cos^2\phi = \dfrac{y^2-k^2}{b^2-k^2}$, $\Delta^2\phi = 1-e^2+e^2\dfrac{y^2}{b^2}$. (43)

42. In the geodesics on the hyperboloid of one sheet, generated by the revolution about Oy of the hyperbola

$$y = \frac{b}{a}\sqrt{(a^2-x^2)}, \tag{44}$$

$$x^2\frac{d\psi}{ds} = k, \tag{45}$$

$$d\psi = \tfrac{1}{2}ke\frac{x^2-\dfrac{a^2}{e^2}}{x^2\sqrt{X}}\,dx^2, \tag{46}$$

where
$$X = x^2-a^2 \,.\, x^2-\frac{a^2}{e^2}\,.\,x^2-k^2, \tag{47}$$

$$e^2 = 1+\frac{b^2}{a^2}, \tag{48}$$

and the parameter
$$v = f\omega_3. \tag{49}$$

When $k^2<a^2$, the geodesic extends over the whole hyperboloid; and

$$\text{(i.)}\quad \infty > x^2 > a^2 > \frac{a^2}{e^2} > k^2, \tag{50}$$

$$s > s_1 > s_2 > s_2 > \sigma, \tag{51}$$

and we put
$$x^2 = m^2(s-\sigma), \tag{52}$$

$$x^2-a^2 = m^2(s-s_1), \tag{53}$$

$$x^2-\frac{a^2}{e^2} = m^2(s-s_2), \tag{54}$$

$$x^2-k^2 = m^2(s-s_3), \tag{55}$$

and find, as before,

$$\psi = \tfrac{1}{2}\frac{P\{\omega_1+(1-f)\omega_3\}}{\sqrt{(s_1-s_3)}}F\phi + I(v), \tag{56}$$

where
$$\sin^2\phi = \frac{a^2-k^2}{x^2-k^2},\quad \cos^2\phi = \frac{x^2-a^2}{x^2-k^2},\quad \Delta^2\phi = \frac{x^2-\dfrac{a^2}{e^2}}{x^2-k^2}. \tag{57}$$

The condition
$$k^2 = \frac{a^2}{e^2} \tag{58}$$

makes $s_2 = s_3$, $\kappa = 0$, and gives the generating lines.

$$\text{(ii.)}\quad \infty > x^2 > a^2 > k > \frac{a^2}{e^2}; \tag{59}$$

s_2 and s_3, k^2 and $\dfrac{a^2}{e^2}$ change places, and we find

$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}}F\phi + I(v), \tag{60}$$

where
$$\sin^2\phi = \frac{a^2-\dfrac{a^2}{e^2}}{x^2-\dfrac{a^2}{e^2}}, \text{ \&c.} \tag{61}$$

$$\text{(iii.)} \quad \infty > x^2 > k^2 > a^2 > \frac{a^2}{e^2}, \tag{62}$$

the geodesic extends from the circle of radius k to infinity, and

$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}}F\phi + I(v),$$

where
$$\sin^2\phi = \frac{k^2-\dfrac{a^2}{e^2}}{x^2-\dfrac{a^2}{e^2}}, \text{ \&c.} \tag{63}$$

43. On the hyperboloid of two sheets, generated by the revolution of the hyperbola

$$x = \frac{a}{b}\sqrt{(b^2+y^2)} \tag{64}$$

about Ox,
$$y^2\frac{d\psi}{ds} = k, \tag{65}$$

leading to
$$d\psi = \tfrac{1}{2}\frac{kae}{b}\,\frac{1}{y^2}\,\frac{\sqrt{\left(y^2+\dfrac{b^4}{a^2+b^2}\right)}}{\sqrt{(y^2+b^2\,.\,y^2-k^2)}}\,dy^2. \tag{66}$$

We take
$$y^2 = m^2(s-\sigma), \tag{67}$$

$$y^2-k^2 = m^2(s-s_1), \tag{68}$$

$$y^2+\frac{b^4}{a^2+b^2} = m^2(s-s_2), \tag{69}$$

$$y^2+b^2 = m^2(s-s_3), \tag{70}$$

and
$$s > s_1 > \sigma > s_2 > s_3, \tag{71}$$

so that
$$v = \omega_1 + f\omega_3, \tag{72}$$

and
$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}} + I(v), \tag{73}$$

$$\sin^2\phi = \frac{k^2+b^2}{y^2+b^2}, \quad \cos^2\phi = \frac{y^2-k^2}{y^2+b^2}, \quad \Delta^2\varphi = \frac{y^2+\dfrac{b^4}{a^2+b^2}}{y^2+b^2}. \tag{74}$$

44. The parameter obtained by the bisection or trisection of a period will serve as an illustration of the general theory; it was employed in Mr. Dewar's stereoscopic diagrams given here.

For the oblate spheroid, and the parameter

$$v = \tfrac{1}{2}\omega_3, \tag{75}$$

we shall find that the projection of the geodesic can be expressed by

$$x^2 \sin\{2\psi - (1-\kappa)F\phi\} = \sqrt{\left(\frac{1+\kappa}{\kappa}\right)}\sqrt{(x^2-k^2)}, \tag{76}$$

$$x^2 \cos\{2\psi - (1-\kappa)F\phi\} = \sqrt{\left(\frac{a^2}{e^2} - x^2 . a^2 - x^2\right)}, \tag{77}$$

where
$$e = \kappa, \quad \frac{b^2}{a^2} = 1-\kappa, \quad \frac{k^2}{a^2} = \frac{1}{1+\kappa}; \tag{78}$$

and now the apsidal angle

$$\Psi = \tfrac{1}{4}\pi \left\{ (1-\kappa)\frac{K}{\frac{1}{2}\pi} + 1 \right\}. \tag{79}$$

By trial it will be found that

$$\Psi = \tfrac{3}{10}\pi, \quad \text{when} \quad \kappa = \sin 58^\circ, \tag{80}$$

so that a closed geodesic can be constructed with these data.

With the parameter
$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{81}$$

we shall find that the equation of the geodesic on the prolate spheroid may be written

$$y^2 \sin\{2\psi - (1+\kappa)F\phi\} = \sqrt{\left(y^2 - k^2 . y^2 + \frac{b^4}{a^2 e^2}\right)}, \tag{82}$$

$$y^2 \cos\{2\psi - (1+\kappa)F\phi\} = \frac{1-\kappa}{\sqrt{\kappa}} b\sqrt{(b^2-y^2)}, \tag{83}$$

where
$$\frac{b^2}{a^2} = \frac{k^2}{b^2} = 1-\kappa, \quad e^2 = \kappa, \tag{84}$$

with an apsidal angle

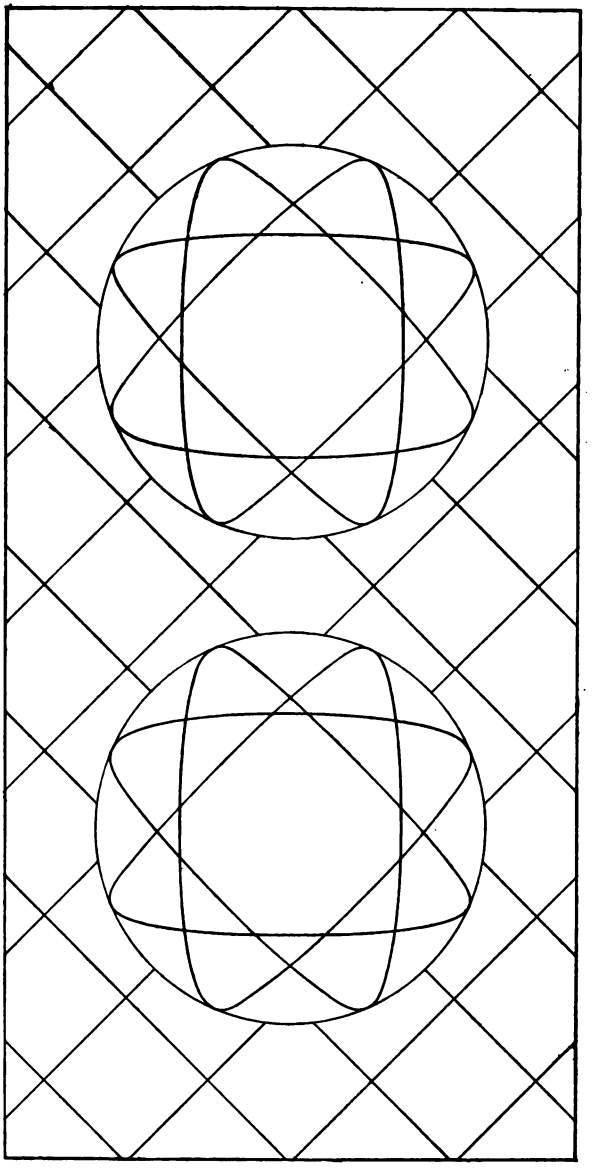
$$\Psi = \tfrac{1}{2}\pi \left\{ (1+\kappa)\frac{K}{\frac{1}{2}\pi} + 1 \right\}. \tag{85}$$

By taking, as in § 21,

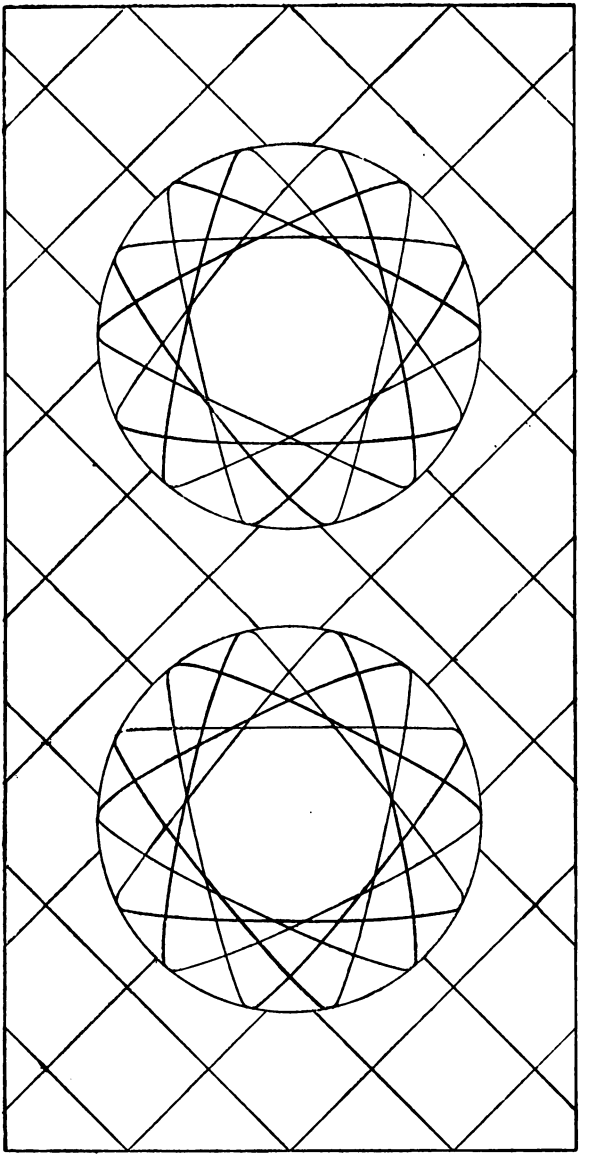
$$\kappa = 0{\cdot}77384 = \sin 50^\circ 42', \tag{86}$$

we make
$$\Psi = \tfrac{4}{5}\pi,$$

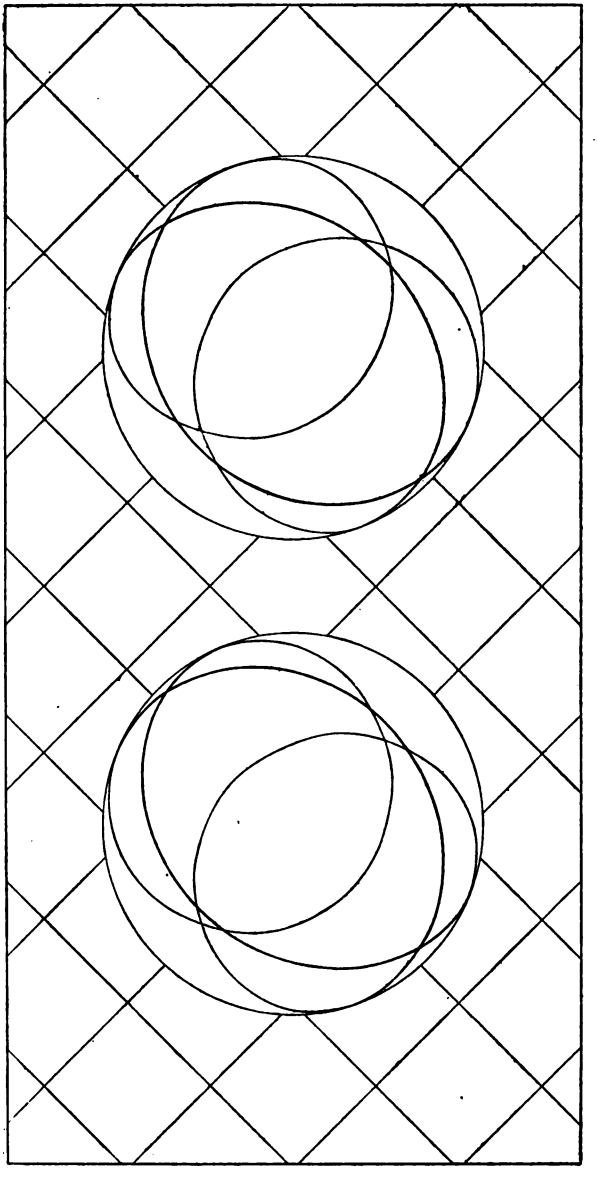
and thus obtain a closed geodesic.



Geodesic on an Oblate Spheroid.



Geodesic on an Oblate Spheroid.



Geodesic on a Prolate Spheroid.

On the hyperboloid of one sheet, with a parameter

$$v = \tfrac{1}{2}\omega_3, \tag{87}$$

the geodesic is given by

$$\text{(i.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(x^2-a^2.x^2-\frac{a^2}{e^2}\right)}, \tag{88}$$

$$x^2 \cos\{2\psi-(1+\kappa)F\phi\} = \surd(x^2-k^2), \tag{89}$$

and

$$e^2 = \frac{1}{\kappa}, \quad \frac{k^2}{a^2} = \frac{\kappa}{1+\kappa}. \tag{90}$$

$$\text{(ii.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \surd(x^2-a^2.x^2-k^2), \tag{91}$$

$$x^2 \cos\{2\psi-(1-\kappa)F\phi\} = a\surd(1+\kappa)\sqrt{\left(x^2-\frac{a^2}{e^2}\right)}, \tag{92}$$

$$\frac{k^2}{a^2} = \frac{a^2}{b^2} = \&c. \tag{93}$$

$$\text{(iii.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \surd(x^2-a^2.x^2-k^2),$$

$$x^2 \cos\{2\psi-(1+\kappa)F\phi\} = a\sqrt{\left(\frac{1+\kappa}{\kappa}\right)}\sqrt{\left(x^2-\frac{a^2}{e^2}\right)}, \tag{94}$$

$$\frac{k^2}{a^2} = \frac{a^2}{b^2} = \frac{1}{\kappa}. \tag{95}$$

On the hyperboloid of two sheets, with a parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{96}$$

we shall find that the geodesic is given by

$$y^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(y^2-k^2.y^2+\frac{b^4}{a^2e^2}\right)}, \tag{97}$$

$$y^2 \cos\{2\psi-(1+\kappa)F\phi\} = b\frac{1-\kappa}{\surd\kappa}\surd(y^2+b^2), \tag{98}$$

$$k^2 = a^2, \quad e^2 = \frac{1}{\kappa}. \tag{99}$$

45. On the surface $$r^2 = x^2+y^2 = a^2 \operatorname{ch}^2\frac{z}{a}, \tag{100}$$

the modified catenoid, the geodesics are given by

$$d\psi = \tfrac{1}{2}k\frac{r^2-(a^2-c^2)}{r^2\surd(r^2-a^2.r^2-k^2.r^2-a^2+c^2)}dr^2, \tag{101}$$

an equation of the same nature as that required for the geodesics on the hyperboloid of one sheet, so that these geodesics are analytically equivalent; this is obvious when we consider that the two surfaces are deformable into each other, without stretching or tearing.

But the catenaries and trajectories are intractable.

Either of these surfaces can also be developed into a skew helicoid, so that the geodesics on this surface are of the same analytical character, which, it may be remarked, is the same as that of Poinsot's herpolhode.

[46. Legendre points out that the integral required in the determination of Φ, the angle of the sector in the development of the surface of the oblique cone on a circular base, is exactly the same as that which gives the angle ψ in the geodesic on a quadric surface of revolution; this is evident from the substitutions

$$m^2(s_2-s) = \frac{f^2\sin^2\omega}{h^2+(r-f\cos\omega)^2}, \tag{102}$$

or

$$m^2(s-s_2) = \frac{h^2+(r-f\cos\omega)^2}{f^2\sin^2\omega}, \tag{103}$$

in his integral for Φ on p. 331, t. I, *Fonctions Elliptiques.*

Then, writing c for $\cos\omega$,

$$m^2(s_1-s) = \frac{\frac{1}{2}f^2}{aa'-h^2-r^2+f^2}\,\frac{\{(a'+a)-(a'-a)\,c\}^2}{h^2+(r-fc)^2}, \tag{104}$$

$$m^2(s-s_3) = \frac{\frac{1}{2}f^2}{aa'+h^2+r^2-f^2}\,\frac{\{(a'+a)\,c-(a'-a)\}^2}{h^2+(r-fc)^2}, \tag{105}$$

or

$$m^2(s-s_1) = \frac{\{(a'+a)\,c-(a'-a)\}^2}{4f^2(1-c^2)}, \tag{106}$$

$$m^2(s-s_3) = \frac{\{(a'+a)-(a'-a)\,c\}^2}{4f^2(1-c^2)}; \tag{107}$$

and with

$$m^2(\sigma-s) = \frac{h^2+r^2-2rfc+f^2}{h^2+(r-fc)^2}, \tag{108}$$

or

$$m^2(s-\sigma) = \frac{h^2+r^2-2rfc+f^2}{f^2(1-c^2)}, \tag{109}$$

in which $h^2+r^2-2rfc+c^2$ is the squared radius vector, and $h^2+(r-fc)^2$ is the squared perpendicular on the tangent of the development; this makes

$$\dot{\Phi} = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma - s} \frac{ds}{\sqrt{S}}, \tag{110}$$

or

$$\Phi = \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s_2 - \sigma} \int \frac{s - s_2}{s - \sigma} \frac{ds}{\sqrt{S}}, \tag{111}$$

exactly as for the determination of ψ in the preceding geodesics.]

On a Regular Rectangular Configuration of Ten Lines. *By* F. Morley. Received May 28th, 1898. Read June 9th, 1898.

1. *The Construction.*—I shall say that a straight line is normal to another when they intersect and are perpendicular. Three lines in space form three pairs, and each pair has a common normal. The three lines, with the three normals, form a rectangular hexagon. The three pairs of opposite sides of this hexagon give three more lines—their common normals. It will be shown that *these last three have one common normal.* Thus, if, starting with three lines, we keep on constructing all possible common normals (excluding the common normals of intersecting lines) we get only ten lines in all, forming a regular configuration in the sense that each line has three normals.

2. *The Points at Infinity.*—Taking five points a, b, c, d, e in space and cutting the ten lines such as ab and the ten planes such as abc by an arbitrary plane, we get a well-known configuration (Fig. 1).

[*Cf.* Cayley's *Math. Papers*, Vol. I., p. 318.]

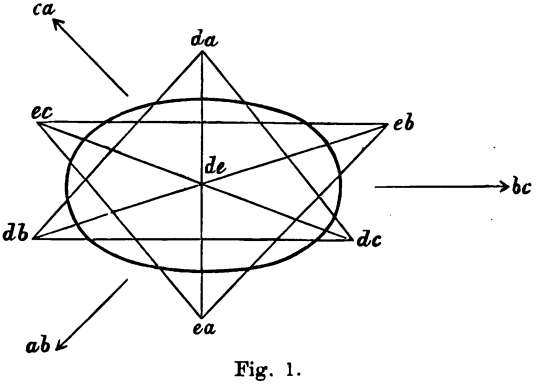


Fig. 1.

Each line *abc* in the plane is the polar of the point *de* with regard to a conic, as appears most easily from the consideration of the space-cubics through *a*, *b*, *c*, *d*, *e* (Reye, *Geometrie der Lage*, Vol. II., p. 273 of the second edition). Three ways of regarding the configuration are to be noticed:—

(i.) Removing a point *de* and its polar *abc*, we have two reciprocal triangles as to the conic, namely, *da*, *db*, *dc* and *ea*, *eb*, *ec*.

(ii.) The complete four-point *ea*, *eb*, *ec*, *ed* contains four of the points and six of the lines; the pairs of lines are conjugate as to the conic. The reciprocal of the four-point as to the conic is the other four lines and the other six points.

(iii.) To a pentagon, such as *ab*, *bc*, *cd*, *de*, *ea*, another *ac*, *ce*, *eb*, *bd*, *da* is both inscribed and circumscribed. The point *ac* is taken arbitrarily on the line *abc*, the second pentagon closing of its own accord.

We are here concerned with the case when the conic is the absolute circle at ∞; the five points *a*, *b*, *c*, *d*, *e* are then *orthocentric*, that is, the line through any two is perpendicular to the plane through the other three.

3. *The Proof.*—Returning to the three lines in space, we suppose them to be (Fig. 2)

$$y = \beta,\ z = -\gamma;\quad z = \gamma,\ x = -\alpha;\quad x = \alpha,\ y = -\beta;$$

and the points at ∞ on them to be

$$ea,\quad eb,\quad ec.$$

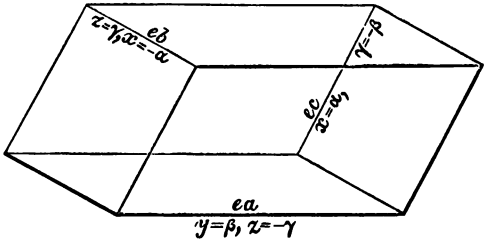


Fig. 2.

The coordinates are oblique; we suppose that the absolute cone is

$$a_1x^2 + b_1y^2 + c_1z^2 + 2fyz + 2gzx + 2hxy = 0,$$

and its reciprocal, the absolute circle, is

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = 0.$$

The coordinates of the ten points at ∞ in Fig. 1 are known from the theory of a conic; thus

$$\begin{aligned} da &\text{ is } A, H, G; \\ bc &\text{ is } 0, g, -h; \\ dc &\text{ is } 1/F, 1/G, 1/H. \end{aligned}$$

The problem of writing the common normals is hereby much simplified. Naming each line in space by its point at ∞, we have for the equations of da, which is the common normal of eb and ec,

$$\frac{z-\gamma}{G}=\frac{x+\alpha}{A}, \quad \frac{x-\alpha}{A}=\frac{y+\beta}{H};$$

for we have merely to pick out from the pencils determining eb and ec the planes through the point

$$x/A = y/H = z/G = \infty .$$

Hence, using the same process, the common normal bc of da and ea, which passes through the point bc, has the equations

$$\frac{A(z-\gamma)-G(x+\alpha)}{-h}=\frac{H(x-\alpha)-A(y+\beta)}{-g},$$

$$\frac{y-\beta}{g}=-\frac{z+\gamma}{h}.$$

Therefore the plane through bc and the point de at ∞ is

$$\frac{(hH+gG)x+\alpha(gG-hH)-gA(z-\gamma)-hA(y+\beta)}{(hH+gG)(GH-AF)} = \frac{h(y-\beta)+g(z+\gamma)}{(hH+gG)F},$$

or, since

$$GH-AF=\Delta . f,$$

$$Fx(gG+hH)-Gy.hH-Hz.gG+Fa(gG-hH)-G\beta.hH + H\gamma.gG+2\Delta(hf\beta-fg\gamma)=0.$$

It is clear that the three such equations, obtained by permuting cyclically x, y, z; f, g, h; F, G, H; and α, β, γ, are consistent, since the left sides have the sum zero. Thus the three planes which should determine the third set of ccmmon normals belong to a pencil; which proves the point.

4. *Restatements.*—As in the plane configuration of § 2, the configuration of ten lines in space can be broken up in different ways. Thus we can state our theorem as follows:—

(i.) The common normals of opposite sides of a rectangular hexagon have a common normal.

(ii.) Four lines in space can be regarded as two pairs in three ways. If the common normals of the two pairs are themselves normal in two of the ways, they are so in the third. This statement of the case was suggested by Mr. Richmond.

(iii.) Two rectangular pentagons can be normal to each other; that is, each side of the one can be normal to a side of the other.

There are in the configuration ten rectangular hexagons, five systems of four lines of the kind just mentioned, and six pairs of mutually normal rectangular pentagons.

Point-Groups in a Plane, and their effect in determining Algebraic Curves. *By* F. S. MACAULAY, D.Sc. Read and received June 9th, 1898.

I. INTRODUCTION.

The following is a continuation of my former paper on "Point-Groups in relation to Curves" in Vol. XXVI. of the *Proceedings*, p. 519. It deals especially with the reduction of point-groups which supply a known number of conditions for an algebraic curve of any order.

The effect of a group of N points in determining an algebraic curve of order n (called hereafter a C_n) need not depend on N and n alone. It may, and often does, happen that the N points do not supply N independent conditions for a C_n, but only a smaller number $N-r_n$. In any case, if the point-group N is given, the number r_n has a definite positive* (integral or zero) value. The extreme case is that in which all the N points lie on a straight line; and we then have $r_n = N-(n+1)$ if $n \leqslant N-1$, and $r_n = 0$ if $n \geqslant N-1$.

For the case in which the N points form the complete intersection of two curves, the values of r_n for all values of n have long been known. Thus, if N consists of the complete intersection of a C_l and C_m, and if n is less than $l+m$, but not less than l or m, then

$$r_n = \tfrac{1}{2}(l+m-n-1)(l+m-n-2),$$

* In this paper the curves are subject to no other conditions than those of passing through points, *i.e.*, they are general algebraic curves through given point-groups in which the parameters or coefficients enter linearly. For linear systems of curves no value of r could be negative; this would not be true for non-linear systems.

by Cayley's theorem; and the value of r_n for *any* value of n may be written in the form

$$\left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right]-\left[\tfrac{1}{2}(m-n-1)(m-n-2)\right]$$
$$-\left[\tfrac{1}{2}(l-n-1)(l-n-2)\right],$$

each pair of square brackets indicating that the product enclosed is only to be retained so long as its individual factors are positive.

But the case in which N is the complete intersection of two curves is only a very special one. It is easy to give other examples. In general, if two point-groups N, N' together make up the complete intersection of two curves C_l, C_m, having no common factor, and if for N there are values of r_n which do not vanish, then for N' there are corresponding values of $r'_{n'}$ which do not vanish, where $n+n'=l+m-3$.

We name the number r_n the *n-ic excess* of the point-group N. It is the number of points which lie on each and every C_n drawn through $N-r_n$ of the N points, *provided these* $N-r_n$ *points are so chosen that their n-ic excess is zero.* Such a choice is evidently always possible,* although it may also be possible to choose $N-r_n$ of the N points whose n-ic excess is not zero.

Corresponding to r_n there is a complementary number q_n, viz., the degree of freedom of a C_n through N, which we name the *n-ic defect* of the point-group N.† This is the number of general points in the

* Such a selection may be made with certainty by choosing the $N-r_n$ points one at a time, each new one being so chosen that a C_n through all those previously chosen does not necessarily pass through it. In this way we must arrive at $N-r_n$ points and no more, since the n-ic excess of N is r_n, neither more nor less.

† [*Note added October 13th.*—The terminology of the theory of point-groups is extensive. This is owing partly to the number of descriptive terms required to distinguish various kinds of point-groups and their special characteristics, and partly to the fact that there are several essentially different ways of approaching the subject. One branch of the subject has been confined almost entirely to Germany, another to Italy, but the branch to which the present paper belongs—that in which the point-groups themselves are subjected to direct operation—has been developed both by English and Continental mathematicians.

The term *regular* (*regolare*) has been applied by Professor G. Castelnuovo to a system of curves in reference to a given point-group when the point-group possesses no *excess* for the general curve of the system. By a slight inversion we may say that a point-group is regular with respect to a general curve of order n when $r_n = 0$. Point-groups which are regular with respect to all general algebraic curves which can be drawn through them I call *general* point-groups, thereby implying only that they are general in their effect in determining algebraic curves.

A convenient English term for point-groups which are not general in the above sense is desirable. The term *constructional* (instead of *point-group of special form*, used in my former paper) seems not inappropriate, since such point-groups can be constructed as the partial (or total) intersection of algebraic curves. The old use of the term *special* was applied to a point-group on a base-curve C_m for which $R-r<p-1$ (p being the deficiency of C_m, R the number of points, and r the multiplicity, of the point-group). The modern use of *special* is, however, applied to any

plane through which a C_n can be drawn which already passes through the N points. Hence, since the N points supply exactly $N-r_n$ conditions for a C_n, and the q_n points, being general, q_n more conditions, we have

$$N-r_n+q_n=\tfrac{1}{2}n(n+3),$$

or

$$N-r_n+q_n+1=\tfrac{1}{2}(n+1)(n+2) \quad \ldots\ldots(1)$$

Hence also

$$r_{n-1}-r_n=n+1-(q_n-q_{n-1}). \quad \ldots\ldots(2)$$

The effect of a point-group N with respect to algebraic curves of all orders is not known unless all the excesses, or all the defects, are known. As we shall find it convenient to regard the subject chiefly from the point of view of the defects, we may suppose at once that the defects q are all given. If $q_n=0$, one, and only one, C_n can be drawn through N. If no C_n can be drawn through N, we assign the value -1 to q_n, the reason for which will be given later (p. 677). It follows that q_n and r_n have definite values for *all* values of n, which always satisfy formula (1) above. It is also shown later that, if all the defects are given, the value of the number N can be deduced.

We may say then that a point-group is fully characterized if all the defects, or all the excesses, are given; and that the point-group has a partially or completely assigned *characterization* according as some or all of the defects, or excesses, are assigned. Two of the most general questions that suggest themselves are: — (i.) What is, and what is not, a possible characterization for a point-group? (ii.) What is the quickest method of arriving at or constructing a point-group with an assigned characterization? These and other questions are answered in the following paper.

The above remarks refer to groups of points which coincide at most in pairs. In the first sections of the paper I have dealt only with ordinary point-groups of this type; but in the footnotes on "Multiple Points,"*

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point-group for which $R-r \leqslant p-1$, that is, to any point-group on C_m which lies on an adjoined C_{m-3}. (*Cf.* Professor Charlotte Angas Scott, "Intersections of Plane Curves," *Bulletin of the American Mathematical Society*, 2nd series, Vol. IV., March, 1898, p. 267). Miss Scott suggests the convenient term *intraspecial* in place of the old term *special*. The connexion between *intraspecial* and *constructional* is expressed as follows:—"An intraspecial point-group on a C_m to which is added an $(i-1)$-point at each i-fold point of C_m is a constructional point-base through which a C_{m-3} can be drawn."

For the meaning of the terms *i-point* and *point-base*, see note on "Multiple Points. I.," below.]

* [*Multiple Points. I.*—It is well known that the number of conditions supplied to a curve by an i-fold point is $\frac{1}{2}i(i+1)$, and that these conditions, when combined with others, may quite possibly not be independent. It is very convenient in the geometrical theory of point-groups to replace, if possible, these

and in the last section of the paper, I have shown how the whole question is capable of generalization.

II. Summary and Deductions.

We use the letter ρ to denote the difference of two successive defects. Thus $\rho_n = q_n - q_{n-1}$. If ρ_l is the first ρ which does not vanish, we have $q_l + 1 = \rho_l$ and $q_p + 1 = \rho_l + \rho_{l+1} + \ldots + \rho_p$ $(p \geq l)$. The differences of the ρ's, or second differences of the defects, are denoted by δ, so that $\delta_l = \rho_l$, $\delta_p = \rho_p - \rho_{p-1}$.

(i.) Writing the general equation of a C_n, referred to two coordinate axes Ox, Oy, in the form

$$C_n \equiv u_0 + u_1 + \ldots + u_n = 0,$$

where u_p is a homogeneous function of x, y of order p, with $p+1$ coefficients, and substituting the coordinates of each of the N points in $C_n = 0$, we have a set of N linear equations for the coefficients of C_n. These equations have always a solution, whether a C_n can be

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$\frac{1}{2}i(i+1)$ conditions by those of passage through $\frac{1}{2}i(i+1)$ points. (*Cf.* pp. 508, 509 of the *Proceedings*, Vol. xxvi.). The theorem which renders such an interchange of conditions practicable is as follows:—"Given a curve C_n, with any number and kind of multiple points, it is always possible to find a curve $C_{n'}$ (n' being either equal to or greater than n) whose coefficients differ only to an infinitely small extent from those of C_n, and such that corresponding to each and every i-fold point A of C_n the curve $C_{n'}$ passes through $\frac{1}{2}i(i+1)$ points chosen arbitrarily and generally about and infinitely near to A." All that is necessary then to effect the change required is to place at each i-fold point $(i > 1)$ a general set of $\frac{1}{2}i(i+1)$ points on an infinitely small scale, and to consider in the place of any curve C_n, with the given multiple points, a proximate curve $C_{n'}$, which passes through all the sets of $\frac{1}{2}i(i+1)$ points, and of which C_n is the limit.

Such a set or *cluster* of $\frac{1}{2}i(i+1)$ points may itself be called a point of order i; we shall therefore call it an *i-point*, reserving the term *i-fold point* for a multiple point of order i on a curve. Thus i-point and i-fold point are practical equivalents; but the one refers to an element of a point-group, and the other to an element of a curve. It should be noticed that an ordinary point is a point of order 1, that is, a 1-point. An i-point is equivalent to $\frac{1}{2}i(i+1)$ simple points. The degree N of a point-group, that is, the total number of simple points to which it is equivalent, is given by $N = \frac{1}{2}\Sigma i(i+1)$, the summation extending to all the points of the group, including those for which $i = 1$.

We also give the name *point-base* (meaning "base of points" = *gruppo base*, Castelnuovo) to a point-group made up of points of assigned orders. In Section vi. we generalize the meaning of this term, and distinguish the point-base here defined by the added epithet *simple*. We shall then, hereafter, only use the term *point-group* in the restricted sense of a group of points which are all of order 1. This is the sense in which it is used throughout the text. The term *point-base* must be distinguished from *base-point*; the latter, however, we shall have no need to use.

The chief importance of this method of dealing with multiple points is that any two curves drawn through an i-point must each have there an i-fold point, and intersect there again in an $(i-1)$-point. Consequently, in the reduction of a point-base, each step leads to a reduction throughout in the orders of the points, while the new points introduced are all of order 1.—*October 13th.*]

drawn through N or not. If the N points lie on a C_n, the general solution of the N equations simply determines a certain number of the coefficients of C_n in terms of the rest, which are left arbitrary. The number so determined is $N-r_n$; and the number left arbitrary is q_n+1, since a C_n through N has still a degree of freedom q_n. If N does not lie on a C_n the solution requires all the coefficients to vanish, and the number determined is $N-r_n=\frac{1}{2}(n+1)(n+2)$, and the number left arbitrary is $q_n+1=0$. *Hence we say that the n-ic defect of a point-group N which does not lie on a C_n is always* -1.

Returning to the case in which a C_n can be drawn through N, the number of arbitrary coefficients in $u_0+u_1+\ldots+u_p$ $(p \leqslant n)$ is q_p+1, by the same reasoning as before; and the number in $u_0+u_1+\ldots+u_{p-1}$ is $q_{p-1}+1$. *Hence the number of arbitrary coefficients in u_p is q_p-q_{p-1}, i.e., ρ_p.* The number ρ_p has therefore a precise analytical interpretation; and, consequently, the properties mentioned below can easily be interpreted analytically. It follows that one limitation to the value of ρ_p is given by $\rho_p \leqslant p+1$, since u_p contains only $p+1$ coefficients in all; but it will be seen later that this limitation disappears when we regard the ρ's from a slightly different point of view.

(ii.) *In order that a point-group N may be a possible one it is necessary and sufficient that the values of the ρ's, after ceasing to be zeros, should consist of continually increasing positive integers, subject to the limitations $\rho_p \leqslant p+1$ and $N+q_p+1 \geqslant \frac{1}{2}(p+1)(p+2)$.*

Both limitations disappear when we regard the number N and the orders of the curves as being given by the values of the ρ's. A still simpler way of enunciating the theorem is :—

A point-group is possible if the second differences of the defects, after once ceasing to be zeros, are positive integers, not including zero; otherwise a point-group is impossible.

It follows from this theorem that if only a C_l can be drawn through N for which the excess of N is not zero, then must $q_l+r_l \leqslant l$. For, in such a case, we have

$$q_{l-1}=-1, \quad r_{l+1}=0;$$

therefore $$\rho_l=q_l-q_{l-1}=q_l+1;$$

and $$r_l-r_{l+1}=l+2-(q_{l+1}-q_l), \quad \text{by (2), p. 675;}$$

therefore $$\rho_{l+1}=l+2-r_l,$$

and $$\rho_{l+1}-\rho_l-1=l-q_l-r_l \geqslant 0.$$

If $q_l+r_l>l$, *and* $q_l \geq 0$, N *must have excess for more than a* C_l *through* N.*

(iii.) *If* $\delta_p \equiv \rho_p - \rho_{p-1} = 1$, *and* $p > \rho_{p-1} > 0$, *then any* C_p *through* N *must contain a fixed constituent curve of order* $p - \rho_{p-1}$.

Not only the order $p-\rho_{p-1}$ of the fixed curve, but the number of points N' on it, is known, and also the full characterization of N' and that of the remainder N'' of the N points.

Thus a point-group N, fully characterized, can be separated into as many constituent point-groups as there are sets of one or more successive δ's equal to 1, each set being preceded and succeeded by one or more δ's greater than 1, together with a remainder.

The points of each constituent (including the remainder when it is not a general point-group) must have a certain number of interconnexions among themselves; but, so far as the characterization of N affects the result, there will not be any connexions between any two of the constituents. Each constituent has its own characterization, and, when its construction has been found, can be placed in any position, without reference to the positions of the other constituents. The sum thus obtained forms the most general point-group N with the assigned characterization. This property of the independence of the constituents, which is not easily apparent by intuition, is here emphasized, since it evidently results in a considerable simplification.†

(iv.) Two theorems suffice for the quickest reduction of a point-group. The first is the theorem mentioned in (iii.), which, in terms of the second differences of the defects, may be expressed as follows:—

(α) *If* $\delta_a = 1$, *then*

$$(\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b) = (\delta_1, \delta_2, \ldots, \delta_{a-1}) + (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \delta_{a+2}, \ldots, \delta_b).$$

The other theorem is

$$(\beta)\ (1^a, \delta_b+1, \delta_{b+1}+1, \ldots, \delta_c+1) + (\delta_c, \delta_{c-1}, \ldots, \delta_b) = I(\Sigma\delta,\ a+\Sigma\delta),$$

where $I(l, m)$ *denotes the complete intersection of a* C_l *and* C_m.

The notation is explained in the next section; but the following remarks will perhaps be intelligible. The characterization of N is fully represented by the numbers

$$\ldots, 0, 0, \delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_{b-1}, \delta_b, 1, 1, 1, \ldots, \textit{ad inf.}$$

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* In the same way it can be proved that if $q_l \geq 0$, $r_{n-2} > 0$, $l < n-2$, then $q_{l-1}+1$ cannot be zero unless $q_{l+1}+1 > 2(q_l+1)$, and r_{n-1} cannot be zero unless $r_{n-3} \geq 2r_{n-2}$.

† This property does not appear to hold in general for a point-base.

Any δ, or any set of successive equal δ's, preceded and succeeded by δ's of higher value is a minimum; and any minimum which reaches its lowest possible value 1 may be called a breaking-point. Theorem (α) exhibits the result after breaking. If there is no breaking-point, theorem (β) shows that from N we can derive a point-group N' such that the δ's of N' are simply the δ's of N each diminished by 1, in reversed order. It is thus evident that all the minima reach breaking-point, some time or other, before the reduction of N has been completed. If the δ's never decrease until the greatest value is reached, and after that never increase, there will be no minimum, and no breaking-point during the whole reduction.

The second differences of the excesses of N are simply δ_1-1, δ_2-1, ..., δ_a-1, ..., δ_b-1, and exhibit the same properties as the second differences of the defects; but the breaking-point is at the value 0 instead of 1.

(v.) *If a C_l is the lowest curve through a point-group N, and a C_{n-2} the highest curve for which the excess of N does not vanish, then the number of the independent interconnexions of the N points, due to the characterization, is*

$$\rho_l(\rho_{l+2}-\rho_{l+1}-1)+\rho_{l+1}(\rho_{l+3}-\rho_{l+2}-1)+\ldots+\rho_{n-2}(\rho_n-\rho_{n-1}-1).*$$

If N breaks up into constituents, *i.e.*, if, for one or more values of p between l and n, $\rho_p-\rho_{p-1}-1=0$, then there are zero terms in the above series, which divide the whole into shorter series. These give

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* This result can be compared with the formula $(q+1)r$ given by Brill and Nöther as the number of conditional equations for the existence of a point-group G^r_R on a C_n. (*Cf.* Benoist's translation of Clebsch, *Leçons sur la Géométrie*, Vol. III., pp. 53 ff.) In the notation adopted above this formula would be written $(q_{n-3}+1)r_{n-3}$. The validity of the formula $(q+1)r$ really rests, however, on the hypothesis that the point-group G^r_R has excess only for a C_{n-3} adjoined to C_n, which requires, as we have seen in (ii.) above, $q+r\leqslant n-3$. If we suppose the excess of the point-group N to be similarly restricted, the number of its interconnexions reduces to

$$\rho_{n-3}(\rho_{n-1}-\rho_{n-2}-1) = (q_{n-3}+1)(n-1-\overline{q_{n-2}-q_{n-3}})$$
$$= (q_{n-3}+1)(r_{n-3}-r_{n-2}) = (q_{n-3}+1)r_{n-3},$$

so that, on this supposition, the two results agree. If $q+r>n-3$, the formula $(q+1)r$ is not in general valid.

It is easy to show that on a non-hyperelliptic curve in which $n>p>1$ the condition $q+r\leqslant n-3$ is necessarily satisfied. For, by the Riemann-Roch equations, $q+r=p-1-(R-2r)$, and, by Bertini's addition to Clifford's theorem, $R-2r>0$. Hence $q+r\leqslant p-2$; and $n\geqslant p+1$; therefore $n-q-r\geqslant 3$, *i.e.*, $q+r\geqslant n-3$. (*Cf.* Miss F. Hardcastle, p. 133 of this volume of the *Proceedings*, and references there given.)

the numbers of the interconnexions of the points of the several constituents, the first series corresponding to what has been called the remainder. As already remarked, the different constituents are unconnected with each other.

There is no reason why a certain number of the points should not coincide in pairs. A point-pair determines a direction, and is expressible in terms of three parameters, viz., the direction and two coordinates. If there are D point-pairs, the $2N$ coordinates of the N points thus reduce to $2N-D$ parameters, and the formula of the theorem then gives the number of independent conditional equations satisfied by these $2N-D$ parameters. A C_n through N will in general have fixed tangents at the point-pairs; but one additional condition applied to C_n at any point-pair will cause it to have there a double point, leaving the directions of its tangents free. It is not so clear as to what interpretation should be put upon the formula if some of the N points coincide in threes or more.

III. Preliminary Theorems and Notation.

(i.) *The n-ic excess r_n of any point-group N diminishes as n increases, until it becomes zero.*

For a C_n can be drawn through any $N-r_n-1$ of the N points without passing through (all) the remainder, and a straight line C_1 can be drawn through any one of the N points without passing through any more. Hence a C_{n+1}, viz., $C_n C_1$, can be drawn through any $N-r_n$ of the N points without passing through (all) the remainder; not so a C_n (Note, p. 674). Hence $r_{n+1} < r_n$.

(ii.) (a) *If a C_p through N is necessarily degenerate, then one constituent of C_p must be fixed.*

Choose any q_p general points in the plane. Then there is one and only one C_p through the $N+q_p$ points. Suppose that this C_p breaks up into $C_{p'} C_{p''}$, and let N' of the N points and q' of the q_p points lie on $C_{p'}$, and the remaining N'' of the N points and q'' of the q_p points lie on $C_{p''}$. Then, since the $q'+q''$ points are general, q' must be the p'-ic defect of N', and q'' the p''-ic defect of N''. Hence, if neither q' nor q'' is zero, a $C_{p'}$ can be drawn through N' and any q' of the $q'+q''$ points, and a $C_{p''}$ through N'' and the remaining q'' of the $q'+q''$ points; and this $C_{p'}$ and $C_{p''}$ would make up a second C_p

through the $N+q_p$ points, which is impossible.* Hence q' or q'' is zero; let q' be zero; then q_p is the p''-ic defect of N''. Hence, in whatever new position we choose q_p general points, the C_p through them and the N points will consist of the $C_{p''}$ through the $N''+q_p$ points, and a fixed $C_{p'}$ through the N' points (whose p'-ic defect is zero).

Also any curve lower than a C_p through N must have the same fixed constituent $C_{p'}$.

(β) *If a C_{n-2} is the highest curve for which the excess of N does not vanish, then a proper C_n can be drawn through N.*

For, if C_n is necessarily degenerate, it must have a fixed constituent $C_{n'}$. Also a C_{n-1} can be drawn through all the N points except one (chosen on $C_{n'}$) without passing through the last, since $r_{n-1}=0$. This C_{n-1} cannot have $C_{n'}$ for a constituent, for, if it had, it would pass through the last point. Also a C_1 can be drawn through the last point in any arbitrary direction. Then $C_{n-1}C_1$ is a C_n through N, not having $C_{n'}$ for a constituent. Hence a proper C_n can be drawn through N.

It may be that a C_{n-1} through N is necessarily degenerate, or that there is no C_{n-1} through N; but a C_{n-1} can be drawn through any $N-1$ of the N points.

(iii.) We express the orders of curves in terms of l, m, n, p, the first three being generally fixed, and the last, p, having any value. A C_l is the lowest curve through N, and a C_{n-2} the highest curve for which N has excess; and we suppose, in general, that N lies on a C_{n-2}. A C_m is the lowest curve through N which has not any fixed constituent, and is not fixed as a whole. We always have $l \leqslant m \leqslant n$.

If the suffixes of q, r, ρ, δ are expressed in terms of l, m, n, p, they are to be understood as having explicit reference to the orders of curves; thus q_p is the p-ic defect of N, and $\rho_p = q_p - q_{p-1}$, $\delta_p = \rho_p - \rho_{p-1}$.

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* [*Multiple Points. II.*—This reasoning fails in a special case when we are dealing with a point-base N. This happens when the constituents of the second C_p are a simple rearrangement of those of the first, so that the C_p itself is not changed. One constituent of C_p, say $C_{p'}$, may be assumed to be a proper curve. If then $q' \geqslant 2$, we could interchange 1 of the q' points with 1 of the q'' points, thus obtaining a second set of q' points, and a second $C_{p'}$, which is certainly not a constituent of the original C_p; in this case therefore the theorem will hold, as also when $q'=0$. But if $q'=1$, it might happen that any second $C_{p'}$, was necessarily a constituent of the original C_p. In this case every proper constituent of the general C_p through N must have defect 0 or 1, and the q_p constituents with defect 1 must belong to a fixed pencil of curves. The conclusion is that, *when a C_p through a point-base N is necessarily degenerate, either* (i.) *one constituent of C_p is fixed, and may break up into several parts, while the remaining constituent is a proper curve with defect q_p; or* (ii.) *C_p breaks up into q_p constituents belonging to a fixed pencil of curves, and the remaining constituents of C_p, if any, are fixed absolutely.—October 13th.*]

But if the suffixes are 1, 2, 3, ..., a, $a+1$, ..., b, ..., they refer only implicitly to the orders of curves; q_a is not the a-ic defect of N, but we still have $\rho_a = q_a - q_{a-1}$, $\delta_a = \rho_a - \rho_{a-1}$.

The symbol $(\rho_1, \rho_2, \ldots, \rho_a)$ represents a point-group whose successive ρ's are $\rho_1, \rho_2, \ldots, \rho_a$, every ρ before ρ_1 being zero, and ρ_{b+1} being equal to $\rho_b + 1$ when $b \geq a$. We suppose also that $\rho_1 > 0$, and $\rho_a > \rho_{a-1} + 1$; for, if not, we can simply leave out the ρ's at the beginning and end until these inequalities are established.

The following properties of the point-group $N = (\rho_1, \rho_2, \ldots, \rho_a)$ will be easily seen by supposing it to be the same as $(\rho_l, \rho_{l+1}, \ldots, \rho_n)$, so that $\rho_1 = \rho_l$, $\rho_2 = \rho_{l+1}$, ..., $\rho_a = \rho_n = n+1$ (since $r_n = r_{n-1} = 0$), and $a = n - l + 1$.

(α) ρ_a is the difference of the defects of N for a $(\rho_a - 1)$-ic and $(\rho_a - 2)$-ic.

(β) The highest curve for which N has excess is a $(\rho_a - 3)$-ic.

(γ) The lowest curve through N is a $(\rho_a - a)$-ic, *remembered as the last ρ diminished by the number of the ρ's.*

(δ) The number of the ρ's exceeds by 2 the number of the different orders of curves through N for which N has excess.

(ϵ) The number N is given by

$$N + q_{n-1} + 1 = \tfrac{1}{2} n (n+1),$$

i.e.,

$$N + \rho_l + \ldots + \rho_{n-1} = \tfrac{1}{2} \rho_n (\rho_n - 1),$$

i.e.,

$$N = \tfrac{1}{2} \rho_a (\rho_a - 1) - (\rho_1 + \rho_2 + \ldots + \rho_{a-1}).$$

(ζ) Thus the values of the ρ's in the point-group $N = (\rho_1, \rho_2, \ldots, \rho_a)$ express the *whole* effect of N in the determination of algebraic curves of all orders. The only restrictions on the values of the ρ's are $0 < \rho_1 < \rho_2 \ldots < \rho_a$, and $\rho_{b+1} = \rho_b + 1$ if $b \geq a$.

The point-groups (ρ_1, ρ_2) and (ρ_1), as indicated in (δ), are general, having zero excess for curves of any order which can be drawn through them. This is easily verified. The number N in (ρ_1, ρ_2) is $\frac{1}{2}\rho_2 (\rho_2 - 1) - \rho_1$, and in $(\rho_1) = (\rho_1, \rho_1 + 1)$ is $\frac{1}{2}\rho_1 (\rho_1 - 1)$.

It may also be verified that the point-group $(1, 2, \ldots, a, \rho_b, \rho_{b+1})$ consists of $\frac{1}{2}\rho_{b+1} (\rho_{b+1} - 1) - (1 + 2 + \ldots + a + \rho_b)$ general points on a $(\rho_{b+1} - a - 2)$-ic [*cf.* (γ) and (ϵ) above]. Similarly $(1, 2, \ldots, a, \rho_b)$, or $(1, 2, \ldots, a, \rho_b, \rho_b + 1)$, consists of $\frac{1}{2}\rho_b (\rho_b - 1) - \frac{1}{2} a (a+1)$ general points on a $(\rho_b - a - 1)$-ic. The point-group $(1, 2, \ldots, a)$ is a zero one.

Expressed in terms of the second differences of the defects $(\rho_1, \rho_2, \ldots, \rho_a, \ldots, \rho_b)$ becomes $(\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b)$; and it is easy to change from one expression to the other ($\delta_1 = \rho_1$, $\delta_a = \rho_a - \rho_{a-1}$). All

the δ's before δ_1 are zeros, and all the δ's after δ_b are units,* while $\delta_1 > 0$, $\delta_b > 1$. The only restriction on the values of the δ's is that they are all positive integers, not including zero.

The lowest curve through $(\delta_1, \delta_2, \ldots, \delta_b)$ is of order $l = \rho_b - b$, or

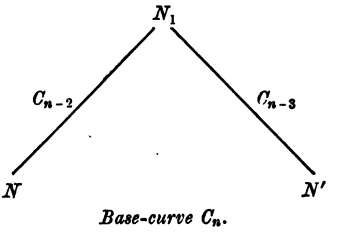
$$l = (\delta_1 - 1) + (\delta_2 - 1) + \ldots + (\delta_b - 1) = \Sigma(\delta - 1).$$

The point-group $(1, 2, \ldots, a, \rho_b, \rho_{b+1}, \ldots, \rho_c)$, when expressed in δ's, is written in the form $(1^a, \delta_b, \delta_{b+1}, \ldots, \delta_c)$; $\delta_b = \rho_b - a$.

IV. Reduction of Point-Groups.

Theorem I.—*A point-group* $(\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 1)$ *can always be found coresidual to a given point-group* $N = (\rho_1, \rho_2, \ldots, \rho_a)$ *on a* $(\rho_a - 1)$*-ic.*

For $(\rho_1, \rho_2, \ldots, \rho_a)$ substitute $(\rho_l, \rho_{l+1}, \ldots, \rho_n)$; then $\rho_n = n + 1$. The highest curve for which N has excess is a C_{n-2}; therefore N lies on a proper C_n [(β), p. 681]. Take this C_n as base-curve. We suppose that $a \geq 3$, otherwise the point-group N is a general one (p. 682); hence $n - 2 \geq l$. Let a C_{n-2} be drawn through N. This will cut C_n again in a finite point-group N_1, since C_n is a proper curve. Also, since the excess of N for a C_{n-2} is not zero, N_1 must, by a known theorem, lie on a C_{n-3}. Let a C_{n-3} through N_1 cut C_n again in a finite point-group N'.†



Base-curve C_n.

Then N, N' are coresidual on C_n, and have identically the same sets of residuals. Also we know that the multiplicity of a series of point-groups on C_n cut out by curves of order p through N is q_p when $p < n$, and $q_p - \frac{1}{2}(p-n+1)(p-n+2)$ when $p \geq n$. The same series is cut out by curves of order $p-1$ through N', and the multiplicity is q'_{p-1} or $q'_{p-1} - \frac{1}{2}(p-n)(p-n+1)$ according as $p < n$ or $p \geq n$. Equating these values,

$$q'_{p-1} = q_p \text{ when } p < n, \quad \text{and} \quad q'_{p-1} = q_p - (p-n+1) \text{ when } p \geq n.$$

* The full expression for $(\delta_1\, \delta_2, \ldots, \delta_b)$ is $(0, \delta_1, \delta_2, \ldots, \delta_b, 1^\infty)$, where 1^∞ stands for 1 repeated *ad inf.*

† *Cf. Proceedings*, Vol. xxvi., p. 525. The N points might necessarily comprise all the N' points. This would be the case if, and only if, N consisted of n points on a straight line, and $N-n$ remaining points having no excess for a C_{n-3}. The N' points would then be identical with the $N-n$ points, but the reasoning would not be affected.

The first of these equations holds not only when $p \geq l$, but also when $p < l$, for then $q'_{p-1} = q_p = -1$. Both equations hold when $p = n-1$.

Taking differences, we have

$$\rho'_{p-1} = \rho_p \text{ when } p < n, \quad \text{and} \quad \rho'_{p-1} = \rho_p - 1 = p \text{ when } p \geq n.$$

The successive values of ρ' are therefore

$$\ldots,\ 0,\ \rho_l,\ \rho_{l+1},\ \ldots,\ \rho_{n-1},\ n,\ n+1,\ n+2,\ \ldots,$$

or

$$\ldots,\ 0,\ \rho_1,\ \rho_2,\ \ldots,\ \rho_{a-1},\ \rho_a - 1,\ \rho_a,\ \rho_a + 1,\ \rho_a + 2,\ \ldots;$$

hence

$$N' = (\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 1),$$

which proves the theorem.

If $\rho_a - 1 = \rho_{a-1} + 1$, then $N' = (\rho_1, \rho_2, \ldots, \rho_{a-1})$; and, if $\rho_a - 1 > \rho_{a-1} + 1$, we can repeat the process on N', thus obtaining the point-group $(\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 2)$. After $\rho_a - \rho_{a-1} - 1$ steps in the reduction we arrive at the point-group $(\rho_1, \rho_2, \ldots, \rho_{a-1})$. In this we leave out all ρ's at the end, if there are any, which exceed the preceding by 1, and continue the reduction. We ultimately arrive at the general point-group (ρ_1, ρ_2), which is a zero point-group if $\rho_1 = 1$ and $\rho_2 = 2$.

This proves that the point-group $N = (\rho_1, \rho_2, \ldots, \rho_a)$ *is a possible one, if* $0 < \rho_1 < \rho_2 < \ldots < \rho_a$.*

It also follows that the point-group N is impossible if $\rho_{b+1} \leq \rho_b > 0$. For, if the point-group is possible, the process of reduction is valid. But if $\rho_{b+1} \leq \rho_b > 0$, then at some stage we arrive at a point-group for which ρ_b is the difference of the defects for curves of order $\rho_b - 2$ and $\rho_b - 1$, and ρ_{b+1} the difference of the defects for curves of order

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* [*Multiple Points. III.*—In the reduction of a point-group which is *general of its kind* (that is, of the kind which has the assigned characterization), the distinguishing feature is that all the derived point-groups are general of *their* kind. It is this that leads to a determinate (not unique) construction for such point-groups. For point-bases the process of reduction is still valid, and so also is the reverse process of construction; but the derived point-bases may not be general of their kind (the kind which possesses a known characterization and consists of points of known order). In such a case the construction is not determined by the reduction alone. Thus if N, in the figure, is a point-base, then for any i-point belonging to N we have an $(i-1)$-point belonging to N_1. The orders of the points of N_1 are all known, and also the characterization (Theorem III.). But from N_1 we have to construct N by passing a C_{n-2} and C_n through it, having i-fold points at the $(i-1)$-points of N_1. Such curves could certainly be drawn through the general point-base of the kind of N_1 if the $(n-2)$-ic defect of N_1 is equal to or greater than Σi, the sum of the orders of all the points of N for which $i > 1$. There is also the condition, in order that the construction may be valid, that the C_n should be a proper curve, or at least should not have any fixed constituent. We return to this again in Note V., p. 688. The question of the generality or speciality of these derived point-bases is not correctly stated in Vol. XXVI., p. 541, § 30.—*October 13th.*]

ρ_b-1 and ρ_b. The corresponding differences of the excesses are then 0 and $\rho_b+1-\rho_{b+1}>0$, by (2), p. 675. Hence the excesses of the derived point-group for curves of order ρ_b-2 and ρ_b-1 are equal, but not zero, which is impossible, by (i.), p. 680.

General Theorems in Reduction.

THEOREM II.—*If* $\rho_a-\rho_{a-1}=1$, *then the point-group*

$$N=(\rho_1, \ldots, \rho_a, \ldots, \rho_b)$$

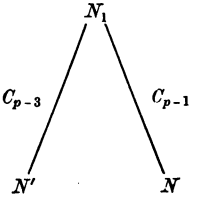
necessarily breaks up into the two independent point-groups

$$(\rho_1, \rho_2, \ldots, \rho_{a-1}) \quad \textit{and} \quad (1, 2, 3, \ldots, \rho_{a-1}, \rho_a, \rho_{a+1}, \ldots, \rho_b).*$$

If $\rho_1=1$, $\rho_2=2$, ..., $\rho_{a-1}=a-1$, then $(\rho_1, \rho_2, \ldots, \rho_{a-1})$ is a zero point-group, and the theorem is nugatory. We shall therefore assume the contrary.

Let $(\rho_1, \ldots, \rho_a, \ldots, \rho_b)$ be the same as $(\rho_l, \ldots, \rho_p, \ldots, \rho_n)$, where $\rho_1=\rho_l$, $\rho_a=\rho_p$, $\rho_p-\rho_{p-1}=1$, $\rho_b=\rho_n=n+1$. We have $p\geq l+1$, since $a\geq 2$.

Then a C_p through N must have a fixed factor; for, if not, a proper C_p can be drawn through N [(α), p. 680], which is different from C_l, since $p>l$. Taking this C_p as base-curve, we can find on it a point-group $N'=N-2p$ coresidual to N, as shown in the figure;† and, by the same reasoning as in the last theorem, we have



Base-curve C_p.

$$q'_{p-4}=q_{p-2}, \quad q'_{p-3}=q_{p-1}, \quad q'_{p-2}=q_p-1;$$

therefore

$$\rho'_{p-3}=\rho_{p-1}, \quad \rho'_{p-2}=\rho_p-1;$$

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* [*Multiple Points. IV.*—It is almost certain that this theorem, except as regards the independence of the constituents, holds in general for a point-base; but it is to be observed:—(i.) that the theorem does not determine in what manner the points of various orders are distributed among the two constituent point-bases, but only the totality of simple points in each; (ii.) that the assigning of orders to the points of N may cause N to break up further than would be required by the characterization alone; and (iii.) that there are limits to the possible values of the orders, combined with the numbers and the characterization, of the points, of which very little is at present known.

It would in many cases be feasible to effect such a distribution of the points of N among N' and N'' that no i-point should belong in part to one and in part to the other; but, if this were not possible, the theorem would not thereby be invalidated. —*October 13th.*]

† A C_{p-1} can be drawn through N, for $p-1\geq l$; and N_1 lies on a C_{p-3}, since N has excess for a C_{p-1}. If N does not lie on a C_{p-2}, then N' does not lie on a C_{p-4}, and $q'_{p-4}=q_{p-2}=-1$.

therefore $$\rho'_{p-3} = \rho'_{p-2} > 0,$$

which is impossible (p. 684).

Then let $C_p = C_{p'} C_{p''}$, where $C_{p'}$ is fixed and $C_{p''}$ has no fixed constituent; and let N' be the number of the N points which lie on $C_{p'}$, and N'' the remainder which lie on $C_{p''}$.

All curves lower than a C_p through N must have the fixed factor $C_{p'}$. Consequently, $q_p, q_{p-1}, q_{p-2}, \ldots$, which are the defects of $N'+N''$ for curves of order $p, p-1, p-2, \ldots$, must be the defects of N'' for curves of order $p'', p''-1, p''-2, \ldots$. Hence the successive differences of the defects of N'' are

$$\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a.$$

But $\rho_a - \rho_{a-1} = 1$; and a $C_{p''}$, to which ρ_a corresponds, has no fixed constituent, by hypothesis; therefore all the ρ's of N'' from ρ_{a-1} increase by units. Hence $$N'' = (\rho_1, \rho_2, \ldots, \rho_{a-1}).$$

This determines the full characterization of N'', and the number N''.

Also, since ρ_{a-1} corresponds to a $C_{p''-1}$, we have

$$p''-1 = \rho_{a-1}-1 = \rho_{p-1}-1, \quad \text{and} \quad p' = p-p'' = p-\rho_{p-1}.$$

The fixed constituent of C_p is therefore of order $p-\rho_{p-1}$.

Again the excess of N'' for a $(\rho_{a-1}-2)$-ic, that is, a $C_{p''-2}$, is zero [(β), p. 682]. Therefore the excess of $N'+N''$ for a C_{p-2} is entirely contributed by N'. The same must be true for all curves higher than a C_{p-2}, whether degenerate, as in the case of a C_{p-2}, C_{p-1}, C_p, or not. But the defects of $N'+N''$ for curves of order $p-2, p-1, \ldots, n$ are

$$q_{p-2}, q_{p-1}, q_p, \ldots, q_n;$$

therefore the defects of N' for curves of the same orders are

$$q_{p-2}+N'', q_{p-1}+N'', q_p+N'', \ldots, q_n+N'',$$

and the differences of these are

$$\rho_{p-1}, \rho_p, \ldots, \rho_n, \quad \text{or} \quad \rho_{a-1}, \rho_a, \ldots, \rho_b.$$

And the differences of successive defects of N' before ρ_{a-1} are

$$1, 2, 3, \ldots, \rho_{a-1}-2, \rho_{a-1}-1,$$

since any curve lower than a C_p through N' has the fixed constituent $C_{p'}$. Hence $$N' = (1, 2, 3, \ldots, \rho_{a-1}, \rho_a, \rho_{a+1}, \ldots, \rho_b).$$

This determines the full characterization of N'. And it can be seen that any two point-groups having the characterizations found for N' and N'', when placed in any positions relatively to one another, make

up a composite point-group with the same characterization as that of

$$N = (\rho_1, \rho_2, \ldots, \rho_a, \ldots, \rho_b).$$

Expressed in terms of the second differences of the defects the theorem is, since $\rho_a = \delta_1 + \delta_2 + \ldots + \delta_a$:—

If $\delta_a = 1$, *then the point-group* $N = (\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b)$ *breaks up into*

$$(\delta_1, \delta_2, \ldots, \delta_{a-1}) + (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \delta_{a+2}, \ldots, \delta_b).$$

Here we may suppose that $\delta_{a+1}, \delta_{a+2}, \ldots, \delta_b$ are all greater than 1. The point-group $(\delta_1, \delta_2, \ldots, \delta_{a-1})$ may, of course, break up further.

The lowest curve through $N = (\delta_1, \ldots, \delta_a, \ldots, \delta_b)$ is a C_l, where $l = \Sigma(\delta - 1)$; and this is a proper curve if $\delta_1, \delta_2, \ldots, \delta_a$ are all equal to 1 and $\delta_{a+1}, \delta_{a+2}, \ldots, \delta_b$ all greater than 1. It follows from what is proved in the next theorem that the lowest curve through N without any fixed constituent is a C_m, where $m = a + \Sigma(\delta - 1)$ if δ_a is equal to 1 and $\delta_{a+1}, \ldots, \delta_b$ are all greater than 1, and $m = \Sigma(\delta - 1) = l$ if $\delta_1, \ldots, \delta_a, \ldots, \delta_b$ are all greater than 1. If $\delta_a = 1$, the fixed curve common to $C_l, C_{l+1}, \ldots, C_{m-1}$ is the lowest curve through

$$N' = (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \ldots, \delta_b),$$

and is therefore of order

$$(\delta_{a+1} - 1) + (\delta_{a+2} - 1) + \ldots + (\delta_b - 1).$$

THEOREM III.—*If* $\delta_b, \delta_{b+1}, \ldots, \delta_c$ *are all positive integers, not including zero, then*

$$(1^a, \delta_b + 1, \delta_{b+1} + 1, \ldots, \delta_c + 1) + (\delta_c, \delta_{c-1}, \ldots, \delta_b) = I(\Sigma\delta, a + \Sigma\delta),$$

$I(l, m)$ *denoting the complete intersection of a* C_l *and* C_m.

This can be deduced from the following theorem (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 526):—

If a C_l *and* C_m *can be drawn through a point-group* N, *cutting again in a finite point-group* N' $(N + N' = lm)$, *then*

$$q'_{l+m-p-3} = r_p - 1 + [\tfrac{1}{2}(l-p-1)(l-p-2)] + [\tfrac{1}{2}(m-p-1)(m-p-2)],$$

square brackets indicating, as before, that the product enclosed is to be retained only if its individual factors are positive. This theorem is true for all values of p from 0 to $l+m-3$, taking $q_p + 1 = 0$ if N does not lie on a C_p.

In applying the theorem we take

$$N = (1^a, \delta_b + 1, \delta_{b+1} + 1, \ldots, \delta_c + 1) = (\delta_l, \delta_{l+1}, \ldots, \delta_m, \ldots, \delta_n),$$

and assume for the moment that a proper C_l and C_m can be drawn through N, where $l = \Sigma\delta$, $m = a + \Sigma\delta$; the assumption being justified at the end. We then have

$$\delta_{l-1} = 0, \quad \delta_l = \delta_{l+1} = \ldots = \delta_{m-1} = 1,$$

$$\delta_m = \delta_b + 1, \ \ldots, \quad \delta_n = \delta_c + 1, \quad \delta_{n+1} = \delta_{n+2} = \ldots = 1.$$

Taking differences in the above theorem, we have

$$q'_{l+m-p-2} - q'_{l+m-p-3} = r_{p-1} - r_p + [l-p-1] + [m-p-1],$$

or

$$\rho'_{l+m-p-2} = p + 1 - \rho_p + [l-p-1] + [m-p-1].$$

Let p diminish from $l+m-3$ to 1, so that $l+m-p-2$ increases from 1 to $l+m-3$. Then

from	$p = l+m-3$ to n,	$\rho'_{l+m-p-2} = p+1-\rho_p$;
„	$p = n$ „ $m-1$,	$\rho'_{l+m-p-2} = p+1-\rho_p$;
„	$p = m-1$ „ $l-1$,	$\rho'_{l+m-p-2} = m-\rho_p$;
„	$p = l-1$ „ 1,	$\rho'_{l+m-p-2} = l+m-p-1-\rho_p$.

Again taking differences, subtracting each equation from the next succeeding, we have

from	$p = l+m-3$ to $n+1$,	$\delta'_{l+m-p-1} = \delta_p - 1 = 0, 0, 0, \ldots, 0$;
„	$p = n$ „ m,	$\delta'_{l+m-p-1} = \delta_p - 1 = \delta_c, \delta_{c-1}, \ldots, \delta_b$;
„	$p = m-1$ „ l,	$\delta'_{l+m-p-1} = \delta_p = 1, 1, \ldots, 1$;
„	$p = l-1$ „ 2,	$\delta'_{l+m-p-1} = \delta_p + 1 = 1, 1, \ldots, 1$.

Hence the successive values of δ', for ascending orders of curves, are

$$\ldots, 0, 0, \delta_c, \delta_{c-1}, \ldots, \delta_b, 1, 1, 1, \ldots;$$

therefore

$$N' = (\delta_c, \delta_{c-1}, \ldots, \delta_b).*$$

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* [*Multiple Points. V.*—We have given only that form of the theorem which is adapted for the quickest reduction of a given point-group. The general theorem is

$$(\ldots \delta_a, \ldots, \delta_b, \ldots, \delta_c) + (\delta_c - 1, \ldots, \delta_b, \ldots, \delta_a + 1, \ldots) = I(f + \Sigma\overline{\delta - 1}, g + \Sigma\overline{\delta - 1}). \quad \ldots\ldots\ldots(A)$$

Here, in order to be perfectly general, we do not suppose the suffixes $\ldots a \ldots b \ldots c$ to be consecutive positive integers. The δ's of $N' = (\delta_c - 1, \ldots, \delta_b, \ldots, \delta_a + 1, \ldots)$, taken in reversed order, are formed from the δ's of $N = (\ldots \delta_a, \ldots, \delta_b, \ldots, \delta_c)$, in direct order, those up to δ_a being increased by 1, thence up to δ_b being unchanged, and thence up to the last δ_c being diminished by 1; f is the number of the δ's of N which are increased, *i.e.*, the number of the δ's in $\ldots \delta_a$, and g is the number of the δ's of N which are not diminished, *i.e.*, the number of the δ's in $\ldots \delta_a, \ldots, \delta_b$. The δ's may have any values which make both N and N' possible; *i.e.*, in the case of point-groups, any number of the δ's at the beginning of both N and N' may be

Hence also we see that it is possible to draw proper curves of order $\Sigma\delta$ and $a+\Sigma\delta$ through $N=(1^a, \delta_b+1, \ldots, \delta_c+1)$; for two proper curves of such order can certainly be drawn through $N'=(\delta_c, \delta_{c-1}, \ldots, \delta_b)$,

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zeros, but after once ceasing to be zeros they must be positive integers, excluding zero.

The specially important case in which $f=0$ gives

$$(\ldots \delta_b, \ldots, \delta_c)+(\delta_c-1, \ldots, \delta_b, \ldots)=I(\Sigma\overline{\delta-1},\ g+\Sigma\overline{\delta-1}), \quad \ldots\ldots\ldots\ldots(B)$$

where g is the number of the δ's in N which remain unchanged in N'.

Taking g equal to the whole number of the δ's in N, we have

$$(\delta_1, \ldots, \delta_c)+(\delta_c, \ldots, \delta_1)=I(\Sigma\overline{\delta-1}, \Sigma\delta). \quad \ldots\ldots\ldots\ldots(C)$$

If the two curves of order $\Sigma(\delta-1)$, $\Sigma\delta$ touch at all the points where they meet, and if N is the point-group formed by all the points of contact (each counted once only), then N' coincides with N; hence the series $\delta_1, \ldots, \delta_c$ is unaltered when reversed. The general case, as may be seen from (A), is almost as simple. In order that a point-group N may be such that two curves can touch at all the N points, without further intersection, then either $N=(\delta_1, \delta_2, \ldots, \delta_2, \delta_1)$, as above, or

$$(\delta_1, \ldots, \delta_{a-1}, \delta_a, \ldots, \delta_a, \delta_{a-1}+1, \ldots, \delta_1+1), \quad \text{or} \quad (\delta_1, \ldots, \delta_a, \delta_a+1, \ldots, \delta_1+1),$$

or one of the two last reversed. It is very remarkable that we can apparently assume any characterization for N, provided it comes under one of these forms and gives the correct value of the number N, without increasing the total number of independent interconnexions of the N points, that is, without increasing the specialization of N.

{For, let C_n, $C_{n'}$ $(n \leqslant n')$ be two curves which cut altogether in two point-groups N, N' having any the same characterization, and let k be the number of independent interconnexions of either point-group due to this characterization. The number of points which can be chosen at will on a given C_n which form part of a point-group on C_n with the same characterization as N is $N+r_n-k$ (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 529). If therefore a $C_{n'}$ can be drawn through N which touches any C_n through N at $N+r_n-k$ of the N points, it will touch it at the remainder. But a $C_{n'}$ can be drawn through N touching C_n at $N-\frac{1}{2}(n-1)(n-2)$ of the N points. Hence the number of conditions that a $C_{n'}$ can be drawn touching C_n at all the N points is

$$(N+r_n-k)-(N-\tfrac{1}{2}\overline{n-1}\,.\,\overline{n-2})=r_n-k+\tfrac{1}{2}(n-1)(n-2)=N-k+q_n+1-3n.$$

Hence, since the degree of freedom of C_n accounts for q_n of the conditions, it follows that $N-k-3n+1$ is the number of conditions to be satisfied by the N points; and to this we can now add the k conditions due to the characterization. Thus the total number is $N-3n+1$, which remains unaltered, whatever the assumed characterization may be.}

The properties expressed by (A), (B), (C) hold equally for point-groups and point-bases; and (B), (C) have applications especially to the latter. What is required to complete the theory in regard to point-bases is, first, to determine the limits of possibility of the δ's as depending on the assigned orders of the points, and, second, to show how to deal with specialized derived point-bases when they cannot be excluded, as in the case of any nine 2-points which lie on a proper sextic.

Whether we are given the characterization of a point-group, or of a point-base, we know the order $\Sigma(\delta-1)$ of the lowest curve which passes through it. Also in the case of a point-group we know the order of the lowest curve without fixed constituents which passes through it; but we do not know it at present with any certainty for a given point-base. Hence for a point-base we have to use (B) in the place of the theorem in the text; and we can only apply (B) by way of trial, for we do not know the lowest value of g which will make the derived point-base N' possible.

We give now an example of a point-base which does not contain any points of

since N' has no excess for a curve of order $\Sigma\delta - 2$ [*cf.* (β), p. 682, and (β), p. 681]; and these determine N, and pass through it.

We add an example of the reduction of a point-group. Take the

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order 1. Very few such examples of a constructional kind are known, the best known being that of the nine 2-points on a proper sextic. Constructional point-bases which include 1-points can be obtained in any number by the methods we have described; not so those which do not include 1-points. (*Cf.* Cayley, *Proc. Lond. Math. Soc.*, Vol. III., p. 197; *Collected Works*, Vol. VII., p. 254.) Consider the point-base N formed by lm 2-points, situated at the lm points in which a C_l and C_m intersect, these lm points being all finitely separated. We know the form of the equation of the general algebraic curve through the point-base, viz.,

$$C_l^2 S + C_l C_m S' + C_m^2 S'' = 0;$$

and we can thence deduce the characterization. The result is

(i.) $N = (3, 3, \ldots \text{repeated } l \text{ times}) = (3^l)$, if $l = m$;

(ii.) $N = (1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l})$, if $l < m$, $2l > m$;

(iii.) $N = (1^{m-l}, 2^l, 1^{m-2l}, 2^l)$, if $2l \leqslant m$.

Suppose now that all that is given with respect to N is its characterization, viz. that in (i.), (ii.), or (iii.), and the fact that N is made up of lm separate 2-points. (The degree $N = 3lm$.) We shall consider first the application of (B) to case (ii.). The lowest curve through N is of order $\Sigma(\delta - 1) = 2l$; call it C_{2l}. (If N is constructed as originally supposed, $C_{2l} = C_l^2$.) If N were a point-group, the lowest curve without fixed constituents through it would be of order $m - l + 2l = l + m$. But, as regards the point-base, this is too low a limit; for, since the two curves must intersect in $4lm$ points at the least, the curve without fixed constituents must be at least of order $2m$. Assume then, by way of trial, that a C_{2m} without fixed constituents can be drawn through N, *i.e.*, that the value of g in (B) is $2(m-l)$. Then the first $2(m-l)$ of the δ's in N are to remain unchanged in N', and the rest are to be diminished each by 1. Thus (B) gives

$$(1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l}) + (1^{m-l}, 2^{2l-m}, 2^{m-l}) = I(2l, 2m).$$

Hence $$N' = (1^{m-l}, 2^{2l-m}, 2^{m-l}) = (1^{m-l}, 2^l) = I(l, m).$$

Now it is possible to draw through $N' = I(l, m)$ two curves C_{2l}, C_{2m} which have double points at all the points of N', and which have no common constituent. The point-base N can therefore be reduced in a single step to an unspecialized point-group $N' = I(l, m)$; and the construction thus found for N is the one originally supposed.

This $N' = I(l, m)$ is the smallest derived of N. The next smallest N'' is obtained by drawing a C_{2l} and C_{2m+1} through N. For this, (B) gives

$$(1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l}) + (1^{m-l}, 2^{2l-m-1}, 3, 2^{m-l}) = I(2l, 2m+1).$$

It can be proved that this $N'' = (1^{m-l}, 2^{2l-m-1}, 3, 2^{m-l})$ must be specialized. For N'' is a point-group containing $ln + 2l$ points, and, if general of its kind, lies on a proper C_{l+1} and a proper C_{m+1}. But through N'' a curve C_{2l} can be drawn, having double points at lm of the N'' points. This C_{2l} and the proper C_{l+1} therefore cut in

$$2lm + 2l = 2l(m+1) > 2l(l+1) \text{ points, since } l < m.$$

Hence C_{2l} must have the proper C_{l+1} for a constituent. Thus

$$C_{2l} \equiv C_{l+1} C_{l-1}.$$

But again, since C_{2l} has double points at each of the lm points, C_{l-1} must pass through the lm points; and C_{l-1} has only $l^2 - 1$ points in all in common with the proper C_{l+1}. Thus $l^2 - 1 \geqslant lm$, and $l^2 > lm$, which is not true. Thus N'' must be

point-group, expressed in terms of the second differences of the defects, $N = (4, 4, 6, 1, 1, 5, 1, 3, 7, 2)$. Working upwards to the defects, we find that this is a group of 404 points whose defects fo curves of order 23 to 33 are $-1, 3, 11, 25, 40, 56, 77, 99, 124, 156, 190$.

$$N = (4, 4, 6) + (1^{16}, 5) + (1^{22}, 3, 7, 2) \quad \text{(Theorem II.)}.$$

(α) (i.) $(4, 4, 6) + (5, 3, 3) = I(11, 11)$ (Theorem III.);

(ii.) $(5, 3, 3) + (2, 2, 4) = I(8, 8)$,,

(iii.) $(2, 2, 4) + (3, 1, 1) = I(5, 5)$,,

(iv.) $(3, 1, 1) = (3, 4)$ in first differences

$= 3$ general points (p. 682).

This gives us the construction of the point-group (4, 4, 6). The number of its points is $11^2 - 8^2 + 5^2 - 3 = 79$.

(β) $(1^{16}, 5) = (1, 2, \ldots 16, 21)$ in first differences

$= 74$ general points on a C_4 (p. 682).

(γ) (i.) $(1^{22}, 3, 7, 2) + (1, 6, 2) = I(9, 31)$ (Theorem III.);

(ii.) $(1, 6, 2) = (1, 7, 9)$ in first differences

$= 28$ general points on a C_6 (p. 682).

Thus $(1^{22}, 3, 7, 2)$ is constructed by drawing a C_9 and C_{31} through 28 general points on a C_6 to cut again in $9 \times 31 - 28 = 251$ points.

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specialized; in fact, C_{l+1} cannot be a proper curve. If the construction found above for N is the only solution, N'' consists of the lm points in which C_l, C_m cut, and l point-pairs on a straight line, viz., at the points where any straight line cuts C_l. This straight line is an l-fold tangent to C_{2m+1}.

This reasoning suggests the inference that the *smallest* derived point-base N' of a given point-base N is the one which *is the most likely to be general of its kind.* Hence the importance of discovering the order of the lowest curve without any fixed constituent which passes through a given point-base.

Taking case (i.), $N = (3^l)$, the lowest curve through N is a C_{2l}, as before. If we assume, by way of trial, that C_{2l} has no fixed constituent, (B) gives (taking $g = 0$)

$$(3^l) + (2^l) = I(2l, 2l).$$

Here $N' = (2^l) = I(l, l)$ satisfies the premised conditions, and is general of its kind. The C_{2l} through N is not a proper curve, but has the requisite property that it does not possess any fixed constituent.

Taking case (iii.), $N = (1^{m-l}, 2^l, 1^{m-2l}, 2^l)$, we see that N breaks up, by Theorem II., into

$$(1^{m-l}, 2^l) + (1^{2m-l}, 2^l) = I(l, m) + I(l, 2m).$$

The two constituents of N, $I(l, m)$ and $I(l, 2m)$, are not independent of one another, nor is the second general of its kind. If the first is general of its kind, the second must consist of point-pairs having the same situation as the single points of the first. This gives a correct analysis of N.—*October 13th.*]

V. Number of Interconnexions of a Point-Group.

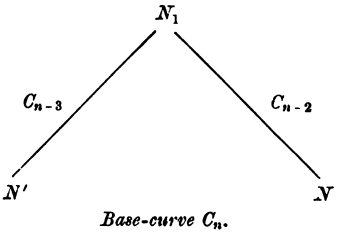
The number of the independent interconnexions of the points of the point-group $(\rho_1, \rho_2, \ldots, \rho_a)$, *due to the characterization, is*

$$\rho_1(\rho_3-\rho_2-1)+\rho_2(\rho_4-\rho_3-1)+\ldots+\rho_{a-2}(\rho_a-\rho_{a-1}-1).$$

To find the number of independent interconnexions of the points of

$$N=(\rho_1, \rho_2, \ldots, \rho_a)=(\rho_i, \rho_{i+1}, \ldots, \rho_n)$$

we go back to Theorem I. Suppose that k is the required number of interconnexions of N, and k' that of N'. We find the value of $k-k'$ by obtaining and equating two different expressions for the least number of parameters in terms of which the base-curve C_n, and the positions of all the points of N and N' upon C_n, can be expressed.



Base-curve C_n.

Taking any two coordinate axes, the least number of parameters in terms of which the positions of the N points in the plane can be expressed is $2N-k$, since k is the number of independent interconnexions of the N points. These $2N-k$ parameters determine the N points; q_n more parameters, and not less, will determine the C_n, since q_n is the defect of N for a C_n; and r'_{n-3} more parameters, and not less, will determine N'. This last result follows from the fact that r'_{n-3} is the multiplicity of N' on C_n.* Thus one of the required expressions is $2N-k+q_n+r'_{n-3}$; and the other is $2N'-k'+q'_n+r_{n-3}$, obtained by starting with N'. Equating these, and noticing that $N+q_n=N'+q'_n$, we have

$$\begin{aligned} k-k' &= (N-r_{n-3})-(N'-r'_{n-3}) \\ &= q'_{n-3}-q_{n-3}, \text{ from (1), p. 675,} \\ &= q_{n-2}-q_{n-3}=\rho_{n-2}=\rho_{a-2}. \end{aligned}$$

Hence, in changing from the point-group $(\rho_1, \ldots, \rho_a)$ to $(\rho_1, \ldots, \rho_{a-1}, \rho_a-1)$, k is diminished by ρ_{a-2}. After $\rho_a-\rho_{a-1}-1$ such steps $(\rho_1, \ldots, \rho_a)$ is reduced to $(\rho_1, \ldots, \rho_{a-1})$, and k is diminished by $\rho_{a-2}(\rho_a-\rho_{a-1}-1)$. But, when $(\rho_1, \ldots, \rho_a)$ has been reduced to the

* A concise statement of the Riemann-Roch theorem is that *the multiplicity of any point-group on a curve* C_n *is equal to the* $(n-3)$*-ic excess of the point-group.* (*Proceedings*, Vol. xxvi., p. 523.)

general point-group (ρ_1, ρ_2), k is diminished to zero. Hence

$$k = \rho_1(\rho_2-\rho_2-1)+\rho_2(\rho_4-\rho_3-1)+\ \dots\ +\rho_{a-2}(\rho_a-\rho_{a-1}-1).*$$

Also, since $k-k' = \rho_{a-2}$, and $N-N' = \rho_a-1$,

therefore $(N-k)-(N'-k') = \rho_a-\rho_{a-2}-1 \geqslant 1$;

therefore $N-k > N'-k' > \ \dots\ \geqslant 0.$

It can be easily proved that

$$N-k = \tfrac{1}{2}(\delta_1-1)\delta_1+\tfrac{1}{2}(\delta_2-1)(\delta_2+2\delta_1)+\ \dots\ +\tfrac{1}{2}(\delta_a-1)(\delta_a+2\delta_{a-1}).$$

VI. Rational Transformation of Point-Bases.†

In rational transformation the whole system of curves of order n which satisfy given conditions transforms into the whole system of curves of another order n' satisfying another set of conditions. If the original conditions are simply those of passage through a given point-base, the transformed conditions are also, so long as n remains fixed, *simply* those of passage through another point-base. But, as the orders n of the original curves increase by units, the orders n' of the transformed curves increase in arithmetical progression, while also the orders of their multiple points may some increase in arithmetical progression, and others remain constant. As n varies, the orders of the points of the transformed point-base also vary.

Rational transformation thus leads to a generalized view of the questions treated above. Instead of investigating the properties in respect to excess and defect of a *simple* point-base, whose points are all of fixed orders, we have to consider these same properties for a

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* [*Multiple Points. VI.*—In the application of this result to point-bases each i-point is to be regarded as a single point. But the reasoning by which the result is obtained fails in the majority of cases, since the proof depends on the use of a slow process of reduction, which would generally cause the derived point-bases to be specialized. The proof can, however, be extended to any reduction of a point-group. Thus it appears that one condition (and probably not the only one) for the correctness of the result, when applied to a given point-base N, is that it should be possible to reduce N by means of a series of unspecialized point-bases; and, for this purpose, as we have seen (Note V., p. 691), the most rapid reduction seems the most likely of any to prove effective.

It seems probable that the result holds for a point-base N so long as it does not exceed twice the number of the points of order 1 contained in N. Further it appears that the correct result, if different from, is less than that found above. Thus in case (i.) of the example in Note V. the value of k, given by the formula $\Sigma\rho_{a-2}(\rho_a-\rho_{a-1}-1)$, is just three times the correct value; and in cases (ii.) and (iii.) the formula gives a value which is *more* than three times too great.—*October 13th.*]

† See footnotes on "Multiple Points" for the meaning of *point-base*, and the applicability to point-bases of the results proved for point-groups.

generalized point-base, including some points of fixed, and others of variable, order. The orders of the curves drawn through the point-base increase with constant difference ν, while the orders of the points of the base increase correspondingly, but each with its own constant difference ι, which varies for different points, and may, in particular, be zero.

In this generalized view the *virtual* number of the conditions supplied for a C_n is $N_n = \frac{1}{2}\Sigma i\,(i+1)$, while the *actual* number is $N_n - r_n$, r_n being the n-ic excess. The n-ic defect, q_n, is still the degree of freedom of a C_n which satisfies the N_n conditions. Instead of formula (1), p. 675, we have

$$N_n - r_n + q_n + 1 = \tfrac{1}{2}(n+1)(n+2).$$

It is evident that all the defects are invariants in rational transformation, since the number of general points through which a C_n, satisfying the N_n conditions, can be drawn is equal to the number of general points through which the transformed $C_{n'}$, satisfying the transformed $N'_{n'}$ conditions, can be drawn, and *vice versâ*. The invariance of the excesses is not so evident; but this can be easily shown by proving it to hold for any quadric transformation. Thus q_n, r_n, ρ_n, δ_n, $N_n - \frac{1}{2}(n+1)(n+2)$ are all invariants, while N_n and n are not. Here $\rho_n = q_n - q_{n-\nu}$, $\delta_n = \rho_n - \rho_{n-\nu}$; and we may further put $\sigma_n = r_{n-\nu} - r_n$, $\epsilon_n = \sigma_{n-\nu} - \sigma_n$.

The invariant $N_n - \frac{1}{2}(n+1)(n+2)$ involves three others, viz., $\nu^2 - \Sigma\iota^2$, $3\nu - \Sigma\iota$, and $n\nu - \Sigma i\iota$. In a simple point-base $\nu = 1$, and all the ι's vanish; hence $\nu^2 - \Sigma\iota^2 = 1$, and $3\nu - \Sigma\iota = 3$; and a generalized point-base derived by rational transformation from a simple one must satisfy these equations. Conversely, if the equations hold for a generalized point-base, it can be rationally transformed into a simple one; for the equations $\nu^2 - \Sigma\iota^2 = 1$, $3\nu - \Sigma\iota = 3$ show that a net of curves C_ν can be described with multiple points of order $\iota, \ldots$ at the points of the point-base. This net rationally transforms the point-base into a simple one; for C_ν transforms to $C_{\nu'}$, consequently $\nu' = 1$, and all the ι''s vanish.

I do not know whether generalized point-bases with similar properties of excess and defect, but having other values than 1 and 3 for $\nu^2 - \Sigma\iota^2$ and $3\nu - \Sigma\iota$, are possible, or not. Assuming them to be possible, the curves C_ν, if they exist, still transform into the curves $C_{\nu'}$, but cannot themselves be used for rational transformation. The numbers 1, 3 are perhaps the lowest possible values of $\nu^2 - \Sigma\iota^2$, $3\nu - \Sigma\iota$ respectively; and, this being so, the curves C_ν certainly exist. If $\nu^2 - \Sigma\iota^2$ were

negative, it would appear that there must be a *superior* limit to the order n of a curve which could satisfy the N_n conditions.

If we write π for the deficiency of C_ν, we have $\delta_n - \epsilon_n = \nu^2 - \Sigma i^2 =$ *constant* number of ordinary points in which two curves C_ν intersect, and $\rho_n + \sigma_n + \pi - 1 = n\nu - \Sigma i_i =$ number of ordinary points in which C_n and C_ν intersect. Reasoning from analogy we should expect 0 to be the least possible value of ϵ_n, corresponding to the breaking-point (p. 679), and $\nu^2 - \Sigma i^2$ to be the least possible value of δ_n.

The Conformal Representation of a Pentagon on a Half Plane.

By Miss M. E. Barwell. Read June 9th, 1898. Received, in revised form, September 15th, 1898.

1. The conformal representation of a rectilinear polygon on a half plane was first attempted by Schwarz and Christoffel, who arrived independently at the same result. They have shown that the area of the w-plane included by a polygon, whose sides do not cross, can be conformally represented by the northern half of the z-plane, the boundary of the polygon corresponding to the axis of real quantities on the z-plane.

The necessary transformation is

$$w = M\int (z-a)^{\alpha-1}(z-b)^{\beta-1}\dots(z-b)^{\lambda-1}\,dz + M',$$

where $a, b, \dots, l$ are the points on the real axis of z corresponding to the angular points of the polygon taken in order, and all lying in the finite part of the z-plane.

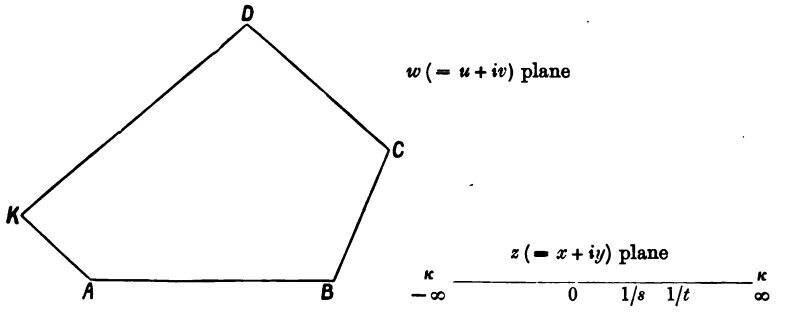
$\alpha\pi, \beta\pi, \dots, \lambda\pi$ are the internal angles of the polygon at the respective points. The constant M' is determined by fixing the origin in the w-plane. Any three of the real quantities $a, \dots, l$ may be chosen arbitrarily, and the remainder must be determined in terms of these three, and the constants of the polygon $\alpha . \beta \dots \lambda$. The case of the quadrilateral is given in Forsyth's *Theory of Functions*, p. 546. There is one unknown quantity besides M to be determined, and the solution involves Gauss' hypergeometric functions.

With the pentagon, there are two unknown quantities, in addition to M, and the necessary equations contain functions analogous to those of Gauss, but satisfying differential equations of the third order (Schäfli, *Crelle*, Vol. LXXVIII.). They may be called "hypergeometric functions of the third order," and they form a special case of the functions investigated by Pochhammer, in his paper on "Hypergeometrische Functionen n^{ter} Ordnung" (*Crelle*, Vol. LXXI.).

Let $\alpha\pi$, $\beta\pi$, $\gamma\pi$, $\delta\pi$, $\kappa\pi$ be the internal angles of the pentagon at A, B, C, D, K respectively. Then

$$\alpha+\beta+\gamma+\delta+\kappa = 3.$$

Also let 0, 1, and ∞ be the homologues in the z-plane of the points A, B, K.



The required transformation is

$$\frac{dw}{dz} = Mz^{\alpha-1}(1-z)^{\beta-1}(1-sz)^{\gamma-1}(1-tz)^{\delta-1},$$

where $\frac{1}{s}$ and $\frac{1}{t}$ correspond to the points C and D, and where

$$1>s>t>0.$$

Suppose $w=0$ at the point A, and that AB is taken as the real axis of w.

Then $$w = M\int_0^z z^{\alpha-1}(1-z)^{\beta-1}(1-sz)^{\gamma-1}(1-tz)^{\delta-1}\,dz,$$

and M must be real.

Integrating along the real axis of z, we obtain the following equations :—

$$AB = M\int_0^1 X\,dx, \tag{1}$$

$$BC.e^{i\pi(1-\beta)} = M\int_1^{1/s} X\,dx, \tag{2}$$

$$CD.e^{i\pi(2-\beta-\gamma)} = M\int_{1/s}^{1/t} X\,dx, \tag{3}$$

$$DK.e^{i\pi(3-\beta-\gamma-\delta)} = M\int_{1/t}^{\infty} X\,dx, \tag{4}$$

$$KA.e^{i\pi(4-\beta-\gamma-\delta-\kappa)} = M\int_{-\infty}^{0} X\,dx, \tag{5}$$

where $X = x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}$.

There are thus five equations to determine M, s, and t; and they must all be consistent by virtue of the relations between the sides and angles of the pentagon. Any three may be selected for the determination of the three unknown quantities.

It is first necessary to reduce equations (1) to (5) to a real form.

Now $$\int_1^{1/s} X\,dx = \int_1^{1/s} x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

$$= e^{\pm i\pi(\beta-1)}\int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx.$$

The lower sign must be taken, for the argument of $1-x$ is $-\pi$, when

$$\frac{1}{s} > x > 1.$$

Hence $$\int_1^{1/s} X\,dx = e^{i\pi(1-\beta)}\int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx.$$

Similarly,

$$\int_{1/s}^{1/t} X\,dx = e^{i\pi(2-\beta-\gamma)}\int_{1/s}^{1/t} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

$$\int_{1/t}^{\infty} X\,dx = e^{i\pi(3-\beta-\gamma-\delta)}\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

and $$\int_{-\infty}^{0} X\,dx = e^{\pm i\pi(\alpha-1)} \int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

$$= e^{i\pi(\alpha-1)} \int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

(since the argument of x is $+\pi$, for $0 > x > -\infty$)

$$= e^{i\pi(2-\beta-\gamma-\delta-\kappa)} \int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx,$$

since $$\alpha+\beta+\gamma+\delta+\kappa = 3.$$

Hence the five equations become

$$AB = M\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx, \qquad \text{(i.)}$$

$$BC = M\int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx, \qquad \text{(ii.)}$$

$$CD = M\int_{1/s}^{1/t} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(1-tx)^{\delta-1}\,dx, \qquad \text{(iii.)}$$

$$DK = M\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(tx-1)^{\delta-1}\,dx, \qquad \text{(iv.)}$$

$$KA = M\int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx. \qquad \text{(v.)}$$

From these equations we can obtain two which will determine s and t, and for this purpose we are at liberty to select any two pairs of equations. For instance, from (ii.) and (iv.),

$$\left.\begin{aligned} DK\int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx \\ = BC\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(tx-1)^{\delta-1}\,dx, \end{aligned}\right\}$$

and, from (i.) and (v.),

$$\left.\begin{aligned} AB\int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx \\ = KA\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx. \end{aligned}\right\} \qquad \text{(A)}$$

2. It is to be observed that the definite integrals in equations (1) to *(5) on p. 697* are hypergeometric functions as defined by Pochhammer

in his paper already quoted (*Crelle*, Vol. LXXI.). He has established as hypergeometric functions of the n^{th} order the integrals typified by

$$\int_g^h (\xi-a_1)^{b_1-1}(\xi-a_2)^{b_2-1}\dots(\xi-a_n)^{b_n-1}(\xi-s)^{\lambda-1}\,d\xi,$$

where g and h may be any two of the quantities $a_1, a_2, \dots, a_n, s, \infty$.

Now, in the typical function of the third order, let

$$\xi=\frac{1}{x}, \quad a_1=0, \quad a_2=t, \quad a_3=1,$$

$$\lambda=\gamma, \qquad b_1=\kappa, \quad b_2=\delta, \quad b_3=\beta;$$

and therefore $3-\lambda-b_1-b_2-b_3=3-\gamma-\kappa-\delta-\beta=\alpha$.

We then obtain the definite integrals given on p. 697.

3. *Expansions of the Integrals on p.* 698.

i. $$\phi_1=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx \qquad (1>s>t>0)$$

is finite, uniform, and continuous, if $\alpha>0$, $\beta>0$, and if s and t have no real positive value between 1 and ∞. Hence the integral can, in the interior of a circle of radius mod $(1-t)$ and centre $s=t$, be developed in a converging series of ascending powers of $s-t$.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

$$=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-tx)^{\delta-1}\{1-tx-(s-t)\,x\}^{\gamma-1}dx$$

$$=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-2}\left\{1-(\gamma-1)\,\frac{s-t}{1-tx}x+\dots\right\}dx$$

$$=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-2}\,dx$$
$$-(\gamma-1)(s-t)\int_0^1 x^{\alpha}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-3}\,dx$$
$$+\dots$$
$$+(-1)^k\frac{(\gamma-1)(\gamma-2)\dots(\gamma-k)}{k!}(s-t)^k$$
$$\times\int_0^1 x^{\alpha+k-1}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-k-2}dx$$
$$+\dots$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} F(2-\delta-\gamma,\ \alpha,\ \alpha+\beta,\ t)$$

$$-(\gamma-1)\frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} F(3-\delta-\gamma,\ \alpha+1,\ \alpha+\beta+1,\ t)(s-t)$$

$$+\ldots$$

$$+(-1)^k \frac{\Gamma(\alpha+k)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+k)} \frac{(\gamma-1)(\gamma-2)\ldots(\gamma-k)}{k!}$$

$$\times F(k+2-\delta-\gamma,\ \alpha+k,\ \alpha+\beta+k,\ t)(s-t)^k$$

$$+\ldots,$$

where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}x+\ldots.$$

Since $$2-\delta-\gamma = \alpha+\beta+\kappa-1,$$

the series may be denoted by $F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\beta,\ 1-\gamma,\ t,\ s-t)$, where the first suffix means that the expansion is that of a hypergeometric function of the *third* order, the second that α is the element which increases by unity from term to term.

ii. $$\phi_2 = \int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

after division by $(1-s)^{\beta+\gamma-1}$, is finite, uniform, and continuous, if $\beta>0$, $\gamma>0$, and if t has no real positive value between s and 1. The integral can therefore be expanded in the neighbourhood of $t=0$ in a converging series of powers of t, and also, in the vicinity of $s=1$, in a converging series of powers of $1-s$.

Let $$x = \frac{1}{1-(1-s)\,\xi};$$

then $$dx = \frac{(1-s)\,d\xi}{\{1-(1-s)\,\xi\}^2},$$

$$1-sx = \frac{(1-s)(1-\xi)}{1-(1-s)\,\xi}, \quad x-1 = \frac{(1-s)\,\xi}{1-(1-s)\,\xi},$$

$$1-tx = 1-\frac{t}{1-(1-s)\,\xi}.$$

Then $$\phi_2 = (1-s)^{\beta+\gamma-1}\int_0^1 \xi^{\beta-1}(1-\xi)^{\gamma-1}(1-\overline{1-s}\,\xi)^{1-\alpha-\beta-\gamma} \times\left\{1-\frac{t}{1-(1-s)\xi}\right\}^{\delta-1} dx$$

$$= (1-s)^{\beta+\gamma-1}\int_0^1 \xi^{\beta-1}(1-\xi)^{\gamma-1}(1-\overline{1-s}\,\xi)^{1-\alpha-\beta-\gamma} \times\left\{1-(\delta-1)\frac{t}{1-1-s\xi}+\ldots\right\} d\xi$$

$$= (1-s)^{\beta+\gamma-1} F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t),$$

in accordance with our notation.

iii. $$\varphi_3 = \int_{1/s}^{1/t} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

after division, by $(s-t)^{\gamma+\delta-1}$, is uniform, finite, and continuous, if $\gamma>0$, $\delta>0$, and if s and t have only values lying between 0 and 1. Thus φ_3 can, in the vicinity of $t=s$, be developed in a converging series of powers of $s-t$, the coefficients of which will involve s.

Let $$\frac{sx-1}{x(s-t)} = \xi;$$

then $$x = \frac{1}{s-(s-t)\xi},\quad dx\,\frac{s-t}{\{s-(s-t)\xi\}^2}d\xi,$$

$$x-1 = \frac{1-s+(s-t)\xi}{s-(s-t)\xi},\quad sx-1 = \frac{(s-t)\xi}{s-(s-t)\xi},$$

$$1-tx = \frac{(s-t)(1-\xi)}{s-(s-t)\xi}.$$

$$\varphi_3 = (s-t)^{\gamma+\delta-1}\int_0^1 \xi^{\gamma-1}(1-\xi)^{\delta-1}\{s-(s-t)\xi\}^{2-\alpha-\beta-\gamma-\delta} \times\{1-s+(s-t)\xi\}^{\beta-1}d\xi$$

$$= (s-t)^{\gamma+\delta-1}s^{\kappa-1}(1-s)^{\beta-1}\int_0^1 \xi^{\gamma-1}(1-\xi)^{\delta-1}\left(1-\frac{s-t}{s}\xi\right)^{\kappa-1} \times\left(1-\frac{s-t}{1-s}\xi\right)^{\beta-1}d\xi$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\delta)}{\Gamma(\gamma+\delta)} s^{\kappa-1}(1-s)^{\beta-1}(s-t)^{\gamma+\delta-1}$$

$$\times\Big[1-\frac{\gamma}{\gamma+\delta}\Big(\frac{\kappa-1}{s}+\frac{\beta-1}{1-s}\Big)(s-t)+\frac{\gamma(\gamma+1)}{(\gamma+\delta)(\gamma+\delta+1)}$$

$$\Big\{\frac{(\kappa-1)(\kappa-2)}{2!\,s^2}+\frac{(\kappa-1)(\beta-1)}{s(s-1)}+\frac{(\beta-1)(\beta-2)}{2!\,(1-s)^2}\Big\}(s-t)^2+\ldots\Big].$$

iv. $$\phi_4 = \int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(tx-1)^{\delta-1}\,dx.$$

When $t=0$ the integral becomes

$$\mathrm{Lt}_{t=0}\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}\,dx,$$

and is therefore infinite and of the order $\alpha+\beta+\gamma-2$, *i.e.*, $1-\kappa-\delta$. Hence, after division, by $t^{\kappa+\delta-1}$, ϕ_4 will be finite, uniform, and continuous, if $\delta>0$, $3-\alpha-\beta-\gamma-\delta>0$, *i.e.*, if $\delta>0$ and $\kappa>0$, and if, moreover, s has no real positive value between 0 and t. Hence $\frac{\phi_4}{t^{\kappa+\delta-1}}$ can be developed in a converging series of powers of t, and also of $1-s$.

Let $$x = \frac{1}{t\xi};$$

then $$dx = -\frac{1}{t\xi^2}\,d\xi,$$

$$x-1 = \frac{1-t\xi}{t\xi}, \quad sx-1 = \frac{s-t\xi}{t\xi}, \quad tx-1 = \frac{1-\xi}{\xi},$$

$$\phi_4 = t^{2-\alpha-\beta-\gamma}\int_0^1 \xi^{2-\alpha-\beta-\gamma-\delta}(1-\xi)^{\delta-1}(1-t\xi)^{\beta-1}(1-t\xi-\overline{1-s})^{\gamma-1}\,d\xi$$

$$= t^{\kappa+\delta-1}\int_0^1 \xi^{\kappa-1}(1-\xi)^{\delta-1}(1-t\xi)^{\beta+\gamma-2}\Big\{1-(\gamma-1)\,\frac{1-s}{1-t\xi}+\ldots\Big\}\,d\xi$$

$$= t^{\kappa+\delta-1}F_{3,\,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ t,\ 1-s).$$

This result might have been obtained from the expansion of ϕ_2 on p. 701, by interchanging β and κ, γ and δ, t and $1-s$.

v. $$\phi_5 = \int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

is uniform, finite, and continuous, if $\alpha>0$, and $3-\alpha-\beta-\gamma-\delta>0$ (*i.e.*, if $\alpha>0$ and $\kappa>0$), and if, moreover, s and t have no real negative

value. Hence, in the neighbourhood of $s=1$, ϕ_5 can be developed in a converging series of powers of $1-s$, and again, when considered as a function of t, it can be expanded in a similar series of powers of $s-t$.

Let $$x=\frac{\xi}{\xi-1}=1+\frac{1}{\xi-1}, \quad dx=\frac{-d\xi}{(1-\xi)^2},$$

$$1-x=\frac{1}{1-\xi}, \quad 1-sx=\frac{1-(1-s)\xi}{1-\xi}, \quad 1-tx=\frac{1-(1-t)\xi}{1-\xi},$$

$$\phi_5=\int_0^1 \xi^{\alpha-1}(1-\xi)^{2-\alpha-\beta-\gamma-\delta}(1-\overline{1-s}\,\xi)^{\gamma-1}(1-\overline{1-t}\,\xi)^{\delta-1}\,d\xi$$

$$=\int_0^1 \xi^{\alpha-1}(1-\xi)^{\kappa-1}(1-\overline{1-s}\,\xi)^{\gamma+\delta-2}\left\{1-(\delta-1)\frac{s-t}{1-\overline{1-s}\,\xi}\xi+\ldots\right\}d\xi$$

$$=F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\kappa,\ 1-\delta,\ 1-s,\ s-t).$$

The two equations on p. 698 for the determination of s and t can now be written in the form

$$\left.\begin{aligned}
&(1)\quad (1-s)^{\beta+\gamma-1}F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t\\
&\qquad=\frac{BC}{DK}t^{\kappa+\delta-1}F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ t,\ 1-s),\\
&(2)\quad F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\beta,\ x-\gamma,\ t,\ s-t)\\
&\qquad=\frac{AB}{KA}F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\kappa,\ 1-\delta,\ 1-s,\ s-t).
\end{aligned}\right\}\quad (B)$$

4. *The Numerical Determination of s and t.*

The questions naturally arise: Are the equations (B) at the end of the last paragraph suitable for the determination of the numerical values of s and t, and why should one pair of the equations (i.) to (v.) on p. 698 be selected rather than another?

With regard to the latter question, let us consider the expansions of the functions ϕ, that is, of the five definite integrals.

ϕ_1 can be expanded in a double series of powers of t and $s-t$;

ϕ_2, after division by $(1-s)^{\beta+\gamma-1}$, in powers of $1-s$ and t;

ϕ_3, after division by $(s-t)^{\beta+\delta-1}$, in powers of $s-t$, the coefficients of which contain s;

ϕ_4, after division by $t^{\delta+\kappa-1}$, in powers of $1-s$ and t;

ϕ_5 in a double series of powers of $s-t$ and $1-s$.

It is in general advantageous to have an equation connecting ϕ_2 and ϕ_4, since they can both be expanded in powers of $1-s$ and t, and for this reason equations (ii.) and (iv.) on p. 698 would be selected. The choice of the other pair would probably depend on the numerical data of the pentagon in question.

Thus an equation connecting t and $1-s$ is

$$(1-s)^{\beta+\gamma-1}.F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t)$$

$$=\frac{BC}{D\kappa}.t^{\kappa+\delta-1}F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ t,\ 1-s).$$

Suppose $\beta+\gamma-1$ and $\kappa+\delta-1$ can each be expressed as the ratio of two integers.

Let $$\beta+\gamma-1=\frac{p'}{p},\quad \kappa+\delta-1=\frac{q'}{q},$$

where p, p', q, q' are integers, which will be in general positive, and let

$$\sigma^p=1-s,\quad \tau^q=t;$$

σ and τ may be supposed small.

We have

$$\sigma^{p'}.F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ \sigma^p,\ \tau^q)$$

$$=\frac{BC}{DK}.\tau^{q'}.F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ \tau^q,\ \sigma^p).$$

This may be written in the form

$$f(\sigma,\tau)\equiv(p'.0)\,\sigma^{p'}+(0.q')\,\tau^{q'}+\ldots=0,$$

where (m, n) denotes the coefficient of $\sigma^m\tau^n$, the series being absolutely convergent for small values of σ and τ.

First, suppose that p' and q' are both unity, *i.e.*, that $\beta+\gamma-1$ and $\kappa+\delta-1$ are aliquot parts of unity. Then, if $f(\sigma,\tau)=0$ be considered as a curve, the origin is an ordinary point, and σ can be expanded in a series of the form

$$c_0\tau+c_1\tau^2+\ldots,$$

convergent for sufficiently small values of σ and τ. Similarly τ can be expanded in a like series of powers of σ.

Next, let one of the quantities p', q' be unity, p' say. Then σ can be expanded in a converging series of integral powers of τ, the exponent of the first term being q'. But τ cannot be expanded in a similar power-series of σ. Thus, in both these cases, we can obtain an expansion for $1-s$ in a series of fractional or integral powers of $1-t$.

Another equation connecting s and t is equation (2) of (B) viz.,

$$F_{3,a}(\alpha+\beta+\kappa-1,\ a,\ \alpha+\beta,\ 1-\gamma,\ t,\ s-t)$$

$$=\frac{AB}{KA}\cdot F_{3,a}(\alpha+\beta+\kappa-1,\ a,\ \alpha+\kappa,\ 1-\delta,\ 1-s,\ s-t).$$

Substituting in this equation the expansion for $1-s$ in terms of t, we have an equation for t in the form of an infinite series.

Now suppose that neither p' nor q' is unity. Then the origin is a singular point of the curve $f(\sigma,\tau)=0$, and σ and τ have respectively p'-ple and q'-ple values. There are, however, p' different convergent expansions for σ of the form

$$\sigma=\Sigma C_r\tau^r,$$

where the exponents r form a series of increasing positive fractional numbers (for the theorem proved in Chrystal's *Algebra*, Vol. II., ch. XXX., (*d*), § 22, for an algebraic function, can be extended to a sufficiently converging infinite series).

Hence for a singular point, by a process such as the use of Newton's parallelogram, corresponding to that employed for an algebraic equation, we can obtain expansions of σ in fractional powers of τ, and consequently expansions of $1-s$ in powers of t. These can be substituted in equation (2) of (B), or in any other of equations (B) which may be convenient.

In the special case of a pentagon which is symmetrical about a line joining A to the middle point of CD, we have

$$\beta=\kappa,\quad \gamma=\delta,\quad AB=AK,\quad BC=DK.$$

The equation (1) of (B) gives $1-s=t$,

and B (2) will be identically satisfied. Combining equations (i.) and (ii.) on p. 698, we have

$$F_{3,a}(\alpha+\beta+\kappa-1,\ a,\ \alpha+\beta,\ 1-\gamma,\ t,\ s-t)$$

$$=\frac{AB}{BC}(1-s)^{\beta+\gamma-1}\cdot F_{3,2-\kappa-\delta}\ 2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t);$$

i.e., $F_{3,2}(\alpha+2\beta-1, \alpha, \alpha+\beta, 1-\gamma, t, 1-2t)$

$$= \frac{AB}{BC} t^{\beta+\gamma+1} F_{3,2-\beta-\gamma}(2-\beta-\gamma, \beta, \beta+\gamma, 1-\gamma, t, t)$$

or $$\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \Big\{ F(2-2\gamma, \alpha, \alpha+\beta, t)$$

$$+(1-\gamma)\frac{\alpha}{\alpha+\beta} F(3-2\gamma, \alpha+1, \alpha+\beta+1, t)(1-2t)$$

$$+\frac{(1-\gamma)(2-\gamma)}{2!} \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\times F(4-2\gamma, \alpha+2, \alpha+\beta+2, t)(1-2t)^2+\dots \Big\}$$

$$= \frac{AB}{BC} t^{\beta+\gamma-1} \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$\times \Big\{ F(2-\beta-\gamma, \beta, \beta+\gamma, t)+(1-\gamma)F(3-\beta-\gamma, \beta, \beta+\gamma, t)\, t$$

$$+\frac{(1-\gamma)(2-\gamma)}{2!} F(4-\beta-\gamma, \beta, \beta+\gamma, t)\, t^2+\dots \Big\},$$

an equation for t in the form of a power-series.

If $$\beta+\gamma = 1,$$

that is, if AB and CD are parallel, the powers of t are all integral.

If the pentagon is regular, the equation becomes

$$F(\tfrac{4}{5}, \tfrac{3}{5}, \tfrac{6}{5}, t)+\tfrac{2}{5}\cdot\tfrac{3}{6}\cdot F(\tfrac{9}{5}, \tfrac{8}{5}, \tfrac{11}{5}, t)(1-2t)+\dots$$

$$= t^{\frac{1}{5}} \{ F(\tfrac{4}{5}, \tfrac{3}{5}, \tfrac{6}{5}, t)+\tfrac{2}{5}\cdot F(\tfrac{9}{5}, \tfrac{3}{5}, \tfrac{6}{5}, t)\, t+\dots \}.$$

On certain Regular Polygons in Modular Network. *By* L. J. Rogers. Received June 7th, 1898; in revised form, August 29th, 1898. Read June 9th, 1898.

1. In connexion with the theory of modular functions, it is proved that a plane may be entirely covered with a network of circles whose interstices are curvilinear triangles equiangular to one another whose angles are all sub-multiples of two right angles. These networks are of three kinds, according as the sum of the angles in each triangle is greater, equal, or less than two right angles. In the

first case, the triangles are limited in number and diverge. They can, moreover, be stereographically projected on to the surface of a sphere, which is thereby divided into equal spherical triangles (see Klein, *Lectures on the Icosahedron*, or Klein-Fricke, *Theorie der elliptischen Modulfunctionen*). In the second case, the network consists of plane triangles, and the circles degenerate into straight lines. In the third kind, the circles all intersect a certain circle orthogonally, and the portions of them internal to this circle form a network which converges to it, while the external portions form a network inverse to the former and re-diverging to infinity (see Klein-Fricke, Vol. I., p. 108). The network of the third kind is the subject of the present paper.

It is easy to see that a property analogous to the stereographic property of the first kind also exists for the third kind. For, if

$$x^2+y^2=1$$

be the orthogonal circle, and

$$a(x^2+y^2+1)+2gx+2fy=0, \quad a'(x^2+y^2+1)+2g'x+2f'y=0$$

be any two circles of the network, then they intersect at an angle

$$\cos^{-1}\frac{gg'+ff'-aa'}{\sqrt{g^2+f^2-a^2}\,\sqrt{g'^2+f'^2-a'^2}},$$

so that, if imaginary lines be drawn whose direction-cosines with respect to three rectangular axes in space are proportional to (g, f, ai) and $(g', f', a'i)$, then planes through the origin perpendicular to them will meet a sphere $x^2+y^2+z^2=1$ in imaginary great circles which intersect in the same real angle as the above circles. The network is thus projected on to an imaginary part of the sphere, and divides it into imaginary spherical triangles. The method, however, has obvious drawbacks, as we cannot re-project any plane section of the sphere back into a real circle unless the normal to the plane has direction cosines of the form $(\lambda, \mu, \nu i)$, where λ, μ, ν are real.

However, the triangles of the network possess a trigonometry with formulæ analogous to those in spherical trigonometry, and which we may easily establish by means of homographic complex transformation, as is indicated by Poincaré in *Act. Math.*, Vol. I., p. 6, &c.

Three portions of network are given in the diagrams accompanying this paper, viz., Fig. 1, in which the angles of every triangle are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$; Fig. 2, in which they are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{9}$; and Fig. 3, in which they are $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{5}$. In general, a network with angles

$\frac{\pi}{\nu_1}$, $\frac{\pi}{\nu_2}$, $\frac{\pi}{\nu_3}$ will be alluded to as (ν_1, ν_2, ν_3), and in the purely geometric treatment of the subject, irrespective of its connexion with modular functions, it is immaterial in what order we write the symbols. We shall, however, always assume that the centre of the orthogonal circle $(x^2+y^2=1)$ is a node of the net.

Nodes of the net will be called ν_1-points, &c., according as the circles meeting there intersect at angles π/ν_1, &c. The origin we shall take as a ν_3-point, and the initial line of coordinates will be one of the diameters of the circle $x^2+y^2=1$ belonging to the network.

If we trace along any circle of the net, it is easy to see the law for the order in which the kinds of nodes follow one another, viz., that a node of even order (*i.e.*, where ν is even) lies between two like nodes, but a node of odd order lies between two unlike.

Thus, if ν_1, ν_2, ν_3 are all odd, they follow one another in rotation.

If, however, one is even (say ν_1), and ν_2, ν_3 are odd, the order is $\dot{\nu}_1\nu_2\nu_3\nu_1\nu_3\dot{\nu}_2$, where dots have the same significance as in circulating decimals. Moreover, this order, or this reversed, is the only possible one.

If two are even (say ν_1, ν_2), while ν_3 is odd, some circles only contain $\dot{\nu}_1\dot{\nu}_2$, while others contain $\dot{\nu}_3\nu_1\nu_3\dot{\nu}_2$.

If the ν's are all even, the orders are $\dot{\nu}_1\dot{\nu}_2$, $\dot{\nu}_2\dot{\nu}_3$, $\dot{\nu}_3\dot{\nu}_1$.

2. It will be convenient to write down for reference the formulæ connecting the sides and angles of any one of these curvilinear triangles which compose the modular network, as indicated by Poincaré in the *Act. Math.*, Vol. I., p. 6. The formulæ in the two following sections will therefore be stated without proof.

The principle involved in the modular function theory is that any circles of the network can be conformally transformed into any other on which the nodes lie in the same order as on the original circle, by the homographic relation $\zeta=(za+b)/(zc+d)$, where z is the complex of any point on the first circle and ζ its copy on the second, and where the orthogonal circle $x^2+y^2=1$ is transformed into itself.

The most general form for such a substitution is

$$\zeta = e^{\varphi i}\frac{z-re^{\theta i}}{zr-e^{\theta i}}, \qquad (1)$$

where r is real.

If a circle cut the unit orthogonal circle in points whose complexes

are x and y, the complex of any point on it is of the form

$$\sqrt{xy}\,\frac{\kappa\sqrt{x}+i\sqrt{y}}{\kappa\sqrt{y}+i\sqrt{x}}, \tag{2}$$

where κ is some real quantity.

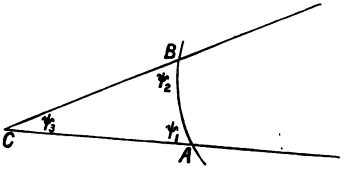
It will be convenient to allude to κ as the *parameter* of the point with respect to the circle.

If κ, λ are the parameters of two points on any circle of the system, then the ratio λ/κ is unaltered for any transformation (1).

The natural logarithm of this ratio will be called the *parametral distance* between the points λ and κ on the circle. It is called by Poincaré the "L" of the arc.

3. The trigonometrical formulæ connecting the angles ψ_1, ψ_2, ψ_3 of a curvilinear triangle with the parametral lengths u_1, u_2, u_3 of the opposite sides can be obtained by supposing the triangle to be made up of two diameters of the orthogonal circle and a circle of the network which lies nearest to the origin.

If CA is the axis of real quantities, the x and y in § 2, (2), are 1



and -1; the parameter of C, the origin, is unity, and the parameter of A is given by

$$CA = \frac{\kappa-1}{\kappa+1} = \frac{e^{u_3}-1}{e^{u_3}+1}$$

By elementary geometry, we have

$$\frac{\sinh u_1}{\sin\psi_1} = \frac{\sinh u_2}{\sin\psi_2} = \frac{\sinh u_3}{\sin\psi_3}. \tag{1}$$

$$\cosh u_1 = \frac{\cos\psi_1 + \cos\psi_2\cos\psi_3}{\sin\psi_2\sin\psi_3}, \tag{2}$$

$$\begin{aligned}\sinh u_1 \sin\psi_2 \sin\psi_3 &= \sqrt{2\cos\psi_1\cos\psi_2\cos\psi_3 + \cos^2\psi_1 + \cos^2\psi_2 + \cos^2\psi_3 - 1} \\ &= n \text{ (say)}, \end{aligned} \tag{3}$$

$$\cosh(u_1+u_2+u_3) = \frac{\cos\psi_1+\cos\psi_2+\cos\psi_3-1}{4\sin\frac{1}{2}\psi_1\sin\frac{1}{2}\psi_2\sin\frac{1}{2}\psi_3}. \tag{4}$$

4. The object of the present paper is to investigate under what conditions it is possible that a network should contain a parametrally regular polygon whose angles are all equal to one of the angles $\frac{\pi}{\nu_1}, \frac{\pi}{\nu_2}, \frac{\pi}{\nu_3}$, and whose sides traced in the same order are conformally equivalent, and are composed of the shortest circular arcs connecting conformal nodes (the angular points of the polygon).

The method employed will be to suppose that two adjacent diametrical lines of the network shall be adjacent sides of the polygon, which may afterwards be transformed so as to have its centre at the origin and thus appear in a symmetrical shape.

Such a polygon, having an angle $\frac{\pi}{\nu}$, I propose to call a *primary* polygon; while a polygon having angles equal to $\frac{m\pi}{\nu}$ will be called *m-ary* polygons. The existence of primary polygons is proved in Klein's *Modulfunctionen*, Vol. I., p. 369, where

$$\psi_1, \psi_2, \psi_3 = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7};$$

also by R. Fricke, *Math. Ann.*, Vol. XLI., p. 458, for the same values of the angles.

Secondary polygons, moreover, exist, as is shown by R. Fricke in *Act. Math.*, Vol. XVII., p. 373, for the network (2, 4, 5), but the aim of the present paper is to prove the existence, and to find the limit to the number, of primary polygons only.

To develop the theory, we need reference to the general treatment of the substitution § 2, (1).

It is shown in Klein's *Modulfunctionen*, Vol. I., p. 163, and in Forsyth's *Theory of Functions*, p. 512, &c., that the substitution in question may be reduced to the form

$$\frac{\zeta - z_1}{\zeta - z_2} = k \frac{z - z_1}{z - z_2},$$

where z_1, z_2, the fixed points of the substitution, are

$$\frac{1}{r} e^{\frac{1}{2}(\theta+\phi)i} \left\{ \cos \tfrac{1}{2}(\theta-\phi) \pm \sqrt{\cos^2 \tfrac{1}{2}(\theta-\phi) - r^2} \right\}, \qquad (1)$$

while
$$k = \frac{1}{1-r^2} \left\{ \sin \tfrac{1}{2}(\theta-\phi) + i \sqrt{\cos^2 \tfrac{1}{2}(\theta-\phi) - r^2} \right\}^2. \qquad (2)$$

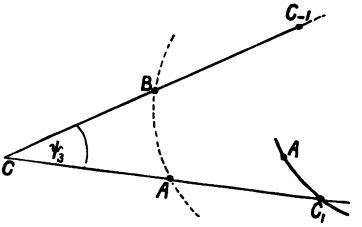
If
$$r^2 < \cos^2 \tfrac{1}{2}(\theta-\phi),$$

this value of k is a complex of form $e^{\psi i}$, and the substitution is *elliptic*, while the fixed points are mutually inverse with respect to $x^2+y^2=1$.

If $$r^2 > \cos^2\tfrac{1}{2}(\theta-\varphi),$$

k is real, while the moduli of the fixed points are unity; that is, the substitution is hyperbolic, and the fixed points lie on the orthogonal circle.

Returning to the subject of the polygons, we see that, if $CA \ldots C_1$ be the initial line and $CB \ldots C_{-1}$ the next diameter in the direction of positive rotation, then the order of nodes from C_{-1} to C must be the same as from C to C_1.



A substitution can then be found to bring C_{-1} to C and C to C_1. Now $CC_1 = CC_{-1}$, for the parametral lengths are equal, and such lengths measured along straight lines from the origin imply equality of actual distance also.

If, then, $CC_1 = \lambda$, the substitution required is

$$\zeta = \frac{z-\lambda e^{\psi_3 i}}{z\lambda - e^{\psi_3 i}} \quad \text{(say } S\text{)}, \tag{3}$$

since it brings $C\,(z=0)$ to $C_1\,(z=\lambda)$ and $C_{-1}\,(z=\lambda e^{\psi_3 i})$ to C.

The same substitution S will convert CC_1 into the nearest circle C_1A_1, which passes through C_1, making an angle of ψ_3 with CC_1 towards C, and a continued application of S produces a series of circles forming a parametrally regular polygon having angles all equal to ψ_3, in fact, a polygon of the type we are seeking.

The only condition for the existence of a closed polygon is that S should be elliptic.

This condition is, by (2), that

$$\lambda^2 < \cos^2\tfrac{1}{2}\psi_3,$$

in which case,
$$k = e^{2\psi i} \quad \text{(say)}$$
$$= \frac{1}{1-\lambda^2}\left(\sin\tfrac{1}{2}\psi_3 + i\sqrt{\cos^2\tfrac{1}{2}\psi_3 - \lambda^2}\right)^2,$$

i.e.,
$$\cos\psi = \frac{\sin\tfrac{1}{2}\psi_3}{\sqrt{1-\lambda^2}}. \tag{4}$$

Conversely, if this requirement is fulfilled, S will be elliptic, and must belong to the group of substitutions which leave the network unaltered. Hence ψ must be a multiple of either ψ_1, ψ_2, or ψ_3.

Let us first suppose the numbers $\nu_1, \nu_2 = \frac{\pi}{\psi_1}, \frac{\pi}{\psi_2}$ are both odd.

Then the period CC_1 is of three segments, and its parametral length is $u_1+u_2+u_3$, so that

$$\lambda = \tanh \tfrac{1}{2}(u_1+u_2+u_3).$$

The relation (4) then becomes

$$\begin{aligned}\cos\psi &= \sin\tfrac{1}{2}\psi_3 \cosh\tfrac{1}{2}(u_1+u_2+u_3)\\ &= \frac{\cos\psi_1+\cos\psi_2+\cos\psi_3-1}{4\sin\tfrac{1}{2}\psi_1\sin\tfrac{1}{2}\psi_2}, \quad \text{by § 3.} \qquad (5)\end{aligned}$$

The condition for a closed polygon is, therefore,

$$\cos\psi_1+\cos\psi_2+\cos\psi_3-1 < 4\sin\tfrac{1}{2}\psi_1\sin\tfrac{1}{2}\psi_2,$$

i.e.,
$$\cos^2\tfrac{1}{2}\psi_3 < (\sin\tfrac{1}{2}\psi_1+\sin\tfrac{1}{2}\psi_2)^2$$

or
$$\cos\tfrac{1}{2}\psi_3 < \sin\tfrac{1}{2}\psi_1+\sin\tfrac{1}{2}\psi_2, \qquad (6)$$

since all the angles are acute.

The condition gives a very limited number of cases :—

Case I.—If $\psi_1 = \psi_2 = \frac{\pi}{3}$,

(6) is satisfied, and $\cos\psi = \cos\psi_3$,

so that for any network $(3, 3, \nu)$ we have a polygon of the type sought. Moreover, since

$$\psi = \psi_3 = \frac{\pi}{\nu},$$

the period of the substitution is ν, so that the polygon has ν sides.

If $\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{5}$,

then $\cos\tfrac{1}{2}\psi_3 < \frac{\sqrt{5}+1}{4} = \cos\frac{\pi}{5}$,

so that $\psi_3 > \frac{2\pi}{5}$.

The only possible value for ψ_3 is therefore $\frac{\pi}{2}$; but, since

$$\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{5} > \pi$$

(the icosahedral case), we exclude this type as not belonging to the network of the third species.

CASE II.—If $\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{7}, \quad \psi_3 = \frac{\pi}{2},$

then $$\cos\psi = \frac{2\cos\frac{\pi}{7} - 1}{4\sin\frac{\pi}{14}} = \frac{\cos\frac{3\pi}{14}}{2\sin\frac{\pi}{7}} = \frac{\sin\frac{2\pi}{7}}{2\sin\frac{\pi}{7}} = \cos\frac{\pi}{7},$$

so that we here have a rectangular heptagon.

However, if $\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{9},$

$$\cos\tfrac{1}{2}\psi_3 < \tfrac{1}{2} + \sin 10^\circ = \cdot 6736 < \cdot 7071 = \cos\frac{\pi}{4},$$

as is seen by tables, so that $\psi_3 > \frac{\pi}{2},$

which is impossible.

A fortiori, if $\psi_1 = \frac{\pi}{3},$

ψ_2 cannot be less than $\frac{\pi}{9}$, so that we can exclude all values of ψ_1 and ψ_2, except $\frac{\pi}{5}$ and $\frac{\pi}{7}$.

Again, when $\psi_1 = \psi_2 = \frac{\pi}{5},$

$$\cos\tfrac{1}{2}\psi_3 < 2\sin 18^\circ = \cdot 6180 < \cos\frac{\pi}{4},$$

so that we may reject this case also, and all other cases in which ν_1 and ν_2 are greater than 5.

We have, therefore, investigated all cases in which ν_1 and ν_2 are odd, and have shown that $(3, 3, \nu)$ and $(3, 7, 2)$ are the only networks for odd ν_1 and ν_2 in which primary polygons occur. Also, when ν_3 is even, ν_1 and ν_2 must be odd, since otherwise we cannot have the same set of nodes on the two lines CA and CB. For along CA, if ν is even, they follow the order $\nu_3\nu_1\nu_3\nu_1\ldots$, and along CB, if ν_2 is odd, the order $\nu_3\nu_2\nu_1\nu_2\nu_3\ldots$.

We will now proceed to the case in which ν_1 is even.

Here there are six segments of the line CC_1 before arriving at C_1 (§ 1), viz., two of each kind, so that the parametral distance CC_1 is $2(u_1+u_2+u_3)$ and $\lambda = \tan(u_1+u_2+u_3).$

This gives $$\sqrt{1-\lambda_3} = \operatorname{sech} 2u = \frac{1}{2\cosh^2 u - 1},$$

so that $\cos\psi = \sin\frac{1}{2}\psi_3 (2\cosh^2 u - 1)$,

which, by § 3, (4),

$$= \frac{c_1c_2c_3 + c_2c_3 + c_1c_2 + c_1^2 + c_2^2 + c_3^2 - c_1 - c_2 - c_3}{2\sin\frac{1}{2}\psi_3 (1-c_1)(1-c_3)}, \qquad (7)$$

where c_1, c_2 c_3 denote the cosines of ψ_1, ψ_2, ψ_3, respectively.

Now it is easy to show, by differentiating $\cos\psi$ partially with respect to ψ_1, ψ_2, ψ_3, that $\cos\psi$ increases when any of these angles decrease, so that we easily obtain a limit as to the smallness of these angles. For instance, when

$$\psi_1 = \psi_3 = \frac{\pi}{3} \quad \text{and} \quad \psi_2 = \frac{\pi}{4},$$

then $$\cos\psi = \frac{3+2\sqrt{2}}{2},$$

which is impossible, and consequently no case is possible unless at least one angle is greater than one of these. Thus the even denominator must be 2. Let, then,

$$\psi_1 = \frac{\pi}{2}.$$

Now, testing for $$\psi_2 = \psi_3 = \frac{\pi}{5},$$

we find $$\cos\psi = \frac{3\sqrt{5}+5}{4},$$

so that one of the odd denominators must be 3.

Case III.—Let, then, $\psi_2 = \frac{\pi}{3}$ (supposing $\psi_2 > \psi_3$).

We now have

$$\cos\psi = \frac{2\cos^2\psi_3 - \cos\psi_3 - \frac{1}{2}}{2\sin\frac{1}{2}\psi_3} = \frac{\cos\frac{5}{2}\psi_3}{2\sin\psi_3}.$$

If $\psi_3 = \frac{\pi}{7}$, then $$\cos\psi = \tfrac{1}{2},$$

which, since the period of the substitution is 3, gives the triangle mentioned in Klein's *Modulfunctionen*, Vol. I., p. 369.

Case IV.—If $\psi_3 = \frac{\pi}{9}$, then

$$\cos\psi = \cos\frac{\pi}{9},$$

which gives a nine-sided polygon.

However, if $\psi_3 = \frac{\pi}{10}$, then

$$\cos\psi = \frac{\sqrt{5}+1}{2\sqrt{2}} > 1,$$

so that ψ_3 cannot be less than $\frac{\pi}{10}$.

Again, we may suppose that

$$\psi_2 < \psi_3 \quad \text{and} \quad \psi_3 = \frac{\pi}{3},$$

so that

$$\cos\psi = \frac{\cos\frac{5}{2}\pi_2}{4\sin\frac{1}{2}\psi_2 \sin\psi_2}.$$

If $\psi_2 = \frac{\pi}{7}$, then

$$\cos\psi = \frac{1}{4\sin\frac{\pi}{14}};$$

but, since

$$\sin^{-1}\tfrac{1}{4} = 14°\,29',$$

approximately, it follows that

$$\sin\frac{\pi}{14} < \sin^{-1}\tfrac{1}{4} \quad \text{and} \quad \cos\psi > 1.$$

Hence ψ_2 cannot be equal to $\frac{\pi}{7}$ or any smaller angle.

We see, then, that the only cases for networks in which only one of the numbers ν_1, ν_2, ν_3 is even, and the corresponding angle is not the angle of the polygon, are (2, 3, 7) and (2, 3, 9).

We now come to the last case in which ν_1, ν_2 are both even.

Here the recurrence of the ν_3-points on the line CA (§ 3) takes place after four segments of parametral length u_2, u_2, u_1, u_1, so that

$$\lambda = \tanh(u_2+u_1);$$

whence

$$\cos\psi = \sin\tfrac{1}{2}\psi_3 \cosh(u_1+u_2).$$

Now

$$\cosh(u_1+u_2) = \cosh u_1 \cosh u_2 + \sinh u_1 \sinh u_2$$

$$= \frac{(c_2c_3+c_1)(c_1c_3+c_2)+n^2}{\sin\psi_1 \sin\psi_2 \sin^2\psi_3}$$

(by the formulæ in § 3)

$$= \frac{c_1c_2c_3+c_1c_2+c_1^2+c_2^2+c_3-1}{2\sin^2\frac{1}{2}\psi_3 \sin\psi_1 \sin\psi_2}.$$

Therefore

$$\cos\psi = \frac{c_1c_2c_3+c_1c_2+c_1^2+c_2^2+c_3-1}{2\sin\frac{1}{2}\psi_3 \sin\psi_1 \sin\psi_2}.$$

Again, in this case, by differentiating, we may show that $\cos\psi$ always increases when any of the angles ψ_1, ψ_2, ψ_3 diminish, so that,

on testing for

$$\psi_1 = \psi_2 = \frac{\pi}{4}, \quad \psi_3 = \frac{\pi}{3},$$

which gives $\cos\psi = \frac{5}{2}$,

we see that no cases are possible unless one of the angles, say

$$\psi_1 = \frac{\pi}{2} \quad \text{and} \quad \psi_2 = \frac{\pi}{4}.$$

Thus $$\cos\psi = \frac{c_2^2 + c_3 - 1}{2\sin\frac{1}{2}\psi_3 \sin\psi_2} = \frac{2\cos\psi_3 - 1}{2\sqrt{2}\sin\frac{1}{2}\psi_3} = \frac{\cos\frac{3}{2}\psi_3}{\sqrt{2}\sin\psi_3}.$$

Case V.—We omit the case $\psi_3 = \frac{\pi}{3}$,

(the octahedral case), since then

$$\psi_1 + \psi_2 + \psi_3 > \pi,$$

and put $$\psi_3 = \frac{\pi}{5},$$

which gives $$\cos\psi = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}.$$

We have then, for the network (2, 4, 5), a polygon of four sides.

Further than this we cannot go, for, even if $\psi_3 = \frac{\pi}{6}$, $\cos\psi = 1$; and, hence, if $\psi_3 < \frac{\pi}{6}$, $\cos\psi > 1$.

Recapitulating, we see that the only possible primary polygons occur in the following networks:—

(3, 3, ν),	containing a	polygon of ν sides for all integral values of ν;
(3, 7, 2),	„	„ heptagon;
(2, 3, 7),	„	„ triangle;
(2, 3, 9),	„	„ polygon of nine sides;
(2, 4, 5),	„	„ quadrilateral.

If ν denote the number of sides in each case, it is evident that, by transforming the network conformally so as to bring a ν-point to the origin, the ν-gon becomes actually symmetrical instead of only parametrally so. However, in the cases (3, 3, ν), (3. 7, 2), and (2, 3, 9) the origin was, in the first instance, a point of the required type, so that the network will contain the actually symmetrical polygon without further transformation.

Subjoined are figures illustrating the triangle (Fig. 1), the nonagon (Fig. 2), and the quadrilateral (Fig. 3) in their symmetrical position. The figure of the heptagon is given in *Math. Ann.*, Vol. XLI., p. 458.

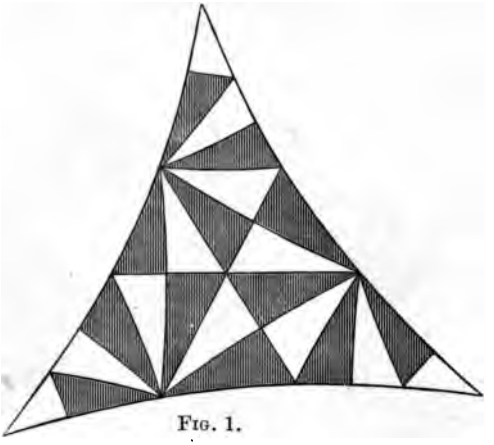


FIG. 1.

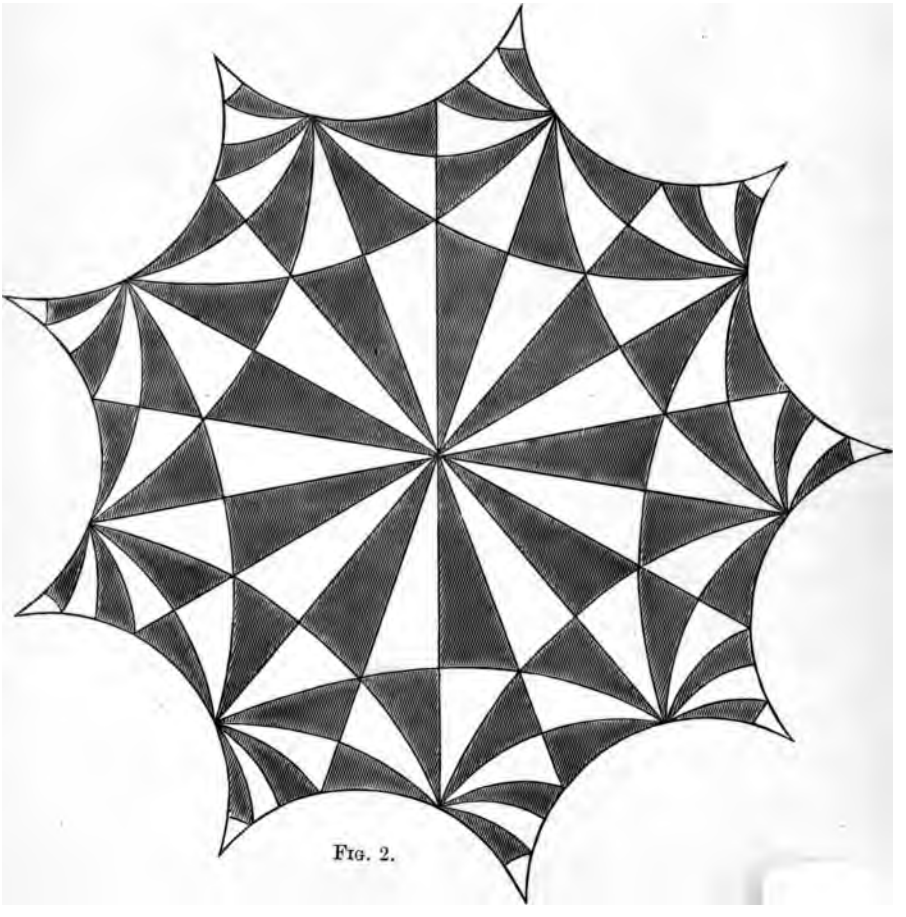


FIG. 2.

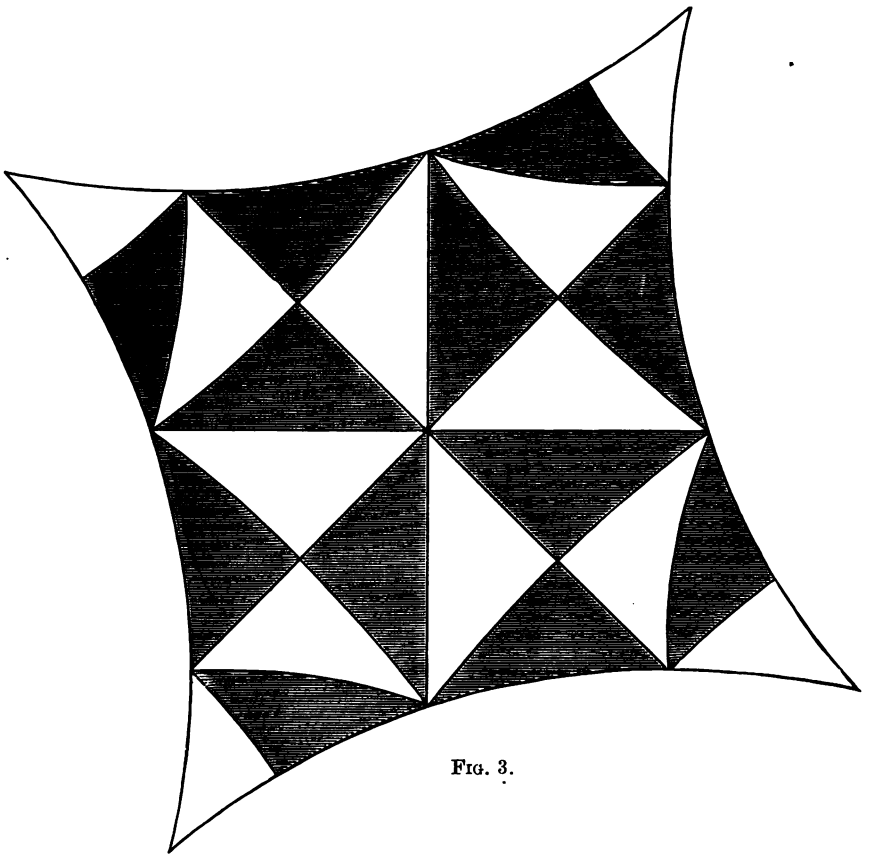


FIG. 3.

5. As an assistance in describing a network, it may be noticed that the centre of every circle, which contains the same order of nodes as the diameters, will be an external node conformal with the origin. Thus the centre $\frac{2xy}{x+y}$, and therefore the inverse $\frac{x+y}{2}$ of the centre of the circle (x, y), will be a ψ_3-point. To prove this, we can first show that, if an elliptic substitution

$$\frac{\zeta-z_1}{\zeta-z_2} = e^{2\psi i}\frac{z-z_1}{z-z_2}, \qquad (1)$$

where z_1, z_2 lie on (x, y), transform (x, y) into (X, Y), then the square of the substitution will transform $\frac{x+y}{2}$ into $\frac{X+Y}{2}$.

Let κ, K be the parameters of z_1 on (x, y) and (X, Y), respectively, so that

$$z_1 = \sqrt{xy}\,\frac{\kappa\sqrt{x}+i\sqrt{y}}{\kappa\sqrt{y}+i\sqrt{x}} = \sqrt{XY}\,\frac{\mathrm{K}\sqrt{X}+i\sqrt{Y}}{\mathrm{K}\sqrt{Y}+i\sqrt{X}},$$

and

$$z_2 = \sqrt{xy}\,\frac{\kappa\sqrt{x}-i\sqrt{y}}{\kappa\sqrt{y}-i\sqrt{x}} = \sqrt{XY}\,\frac{\mathrm{K}\sqrt{X}-i\sqrt{Y}}{\mathrm{K}\sqrt{Y}-i\sqrt{X}}.$$

Now (1) changes x into X, and, substituting these values for z and ζ, we obtain

$$\frac{\mathrm{K}\sqrt{Y}-i\sqrt{X}}{\mathrm{K}\sqrt{Y}+i\sqrt{X}} = e^{2\psi i}\,\frac{\kappa\sqrt{y}-i\sqrt{x}}{\kappa\sqrt{y}+i\sqrt{x}}. \qquad (2)$$

Again, if

$$z = \frac{x+y}{2},$$

we easily get

$$\frac{z-z_1}{z-z_2} = \left(\frac{\kappa\sqrt{y}-i\sqrt{x}}{\kappa\sqrt{y}+i\sqrt{x}}\right)^2;$$

so that, by (2), we see that

$$\frac{\frac{1}{2}(X+Y)-z_1}{\frac{1}{2}(X+Y)-z_2} = e^{4\psi i}\,\frac{\frac{1}{2}(x+y)-z_1}{\frac{1}{2}(x+y)-z_2},$$

which shows that the square of (1) brings $\frac{x+y}{2}$ to $\frac{X+Y}{2}$.

Now, for a diameter, $\frac{x+y}{2}$ is the origin, and, as we arrive at any circle conformal with a diameter by a series of rotations about nodes, the inverses of the centres are successively derived from the origin by the series of the squares of the same rotations, and are therefore all conformal with the origin.

The following presents were made to the Library during the Recess :—

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APPENDIX.

The Society's loss, through the deaths of its members, has been again a large one. For the following notice of Signor Francesco Brioschi we are indebted to documents put at our disposal by Prof. Luigi Cremona.

La sera del 13 mese corrente moriva in Milano, sua città natale, il nostro collega Francesco Brioschi, dopo breve malattia, aggravata forse dalla fiducia inspiratagli dalla costante immunità. La sua morte, del tutto impreveduta, ci ha colpiti del più doloroso stupore: sebbene egli fosse alla vigilia del compimento del suo 73mo anno, a vederlo cosi vivace ed agile, forte e grigio come acciaio, nessuno avrebbe dubitato che la sua operosità non fosse per durare ancora molti anni: tanta era in lui la robustezza della fibra e della mente, che non avevano mai conosciuto stanchezza. Era nato il 22 dicembre 1824, aveva fatto gli studi universitari a Pavia e vi si era laureato ingegnere con onore, nel 1845. Ad innamorarlo vieppiù dell' alta matematica contribui Gabrio Piola, il dotto commentatore delle opere di Lagrange.*

* From the *Resoconto Sommario* (Senato del Regno), Lunedi, 20 Dicembre 1897, the President, for the time, being Prof. Cremona. The rest of the speech treats rather of Brioschi as a public man than as a mathematician.

The following estimate of Brioschi, as a mathematician, is taken from that furnished by Signor Beltrami to *La Perseveranza* of Thursday, December 23rd, 1897.*

Straordinaria versatilità d'ingegno, singolare potenza di assimilazione rapida e profonda, instancabile tenacia in ogni ricerca come in ogni intrapresa, tali furono le doti onde rifulse la mente, davvero eccelsa, di Francesco Brioschi.

Nelle scienze matematiche poi, in cui egli raggiunse più presto la gloria, e gloria solida e duratura, recò un' altra preziosissima dote, quella che or si direbbe una impareggiabile virtuosità, cioè una agilità elegante di forma e di pensiero, la quale, se sgomentava gli impazienti ed i meno provetti, formava l' ammirazione dei dotti e degli studiosi di lena.

Allevato alla scuola di Bordoni nel culto, forse un po' troppo esclusivo, dei metodi lagrangiani, entrò poco appresso con Piola nell' ambiente meno rigido delle ricerche fisico-matematiche di Fourier, di Poisson e di Cauchy. Ma, per quanto grande e feconda fosse allora la produzione matematica francese, che era la sola cui attingessero i pochi studiosi d' Italia, egli intuì ben presto la necessità d' allargare la cerchia delle fonti, estendendola alla produzione delle altre nazioni colte d' Europa, massimamente della Germania e dell' Inghilterra. Fra noi egli fu indubbiamente il primo a mettersi risolutamente per questa via, e ad indirizzarvi quanti allievi e studiosi potè attirare con sè in quest' opera, che può ben dirsi di risanamento, giacchè soltanto per essa cessò quel tal quale ristagno che da lungo tempo pesava sulla scienza italiana, e incominciò quel sempre più attivo e fecondo ricambio intellettuale colla scienza e cogli scienziati di fuori, che fu certamente favorito e promosso dalla fortunata ricostituzione dell' unità nazionale, ma che sarebbe ingiustizia non revocare a lui, per ciò che spetta alle sue prime origini, ben più modeste, ma ben più laboriose.

È incredibile la quantità di lavori che il Brioschi seppe comporre e produrre in luce, nei più svariati indirizzi, non appena si fu rapidamente orientato nel vastissimo campo delle ricerche che occupavano, alla metà di questo secolo, i più valorosi matematici d' Europa. La recente scomparsa del grande Jacobi, col quale il Brioschi aveva

* The same writer furnishes a "Necrologia" to the *Annali di Matematica*, Tomo XXVI., Fasc. 4, pp. 343–347. *Cf.* also "Francesco Brioschi, Cenno Necrologico per Alfredo Capelli," in the *Giornale di Matematiche*, Vol. XXXVI., Jan.–Feb., 1898, p. 51.

tanta affinità di temperamento scientifico, richiamava allora la attenzione sul grande problema delle equazioni dinamiche, e fu questo uno dei primi soggetti a cui egli si rivolse, trattovi anche dalla natura dell' insegnamento che impartiva all' Università di Pavia.

Numerosi ed apprezzatissimi sono i suoi lavori, sia sul problema d' integrazione, sia sulle affini teorie delle equazioni a derivate parziali e delle equazioni isoperimetriche. Nè cessò mai, anche a maggior distanza di tempo, di ritornare su quei primi studii con altre geniali pubblicazioni, come, a cagion d' esempio, con quelle relative all' ellissoide fluido di Dirichlet ed al problema dei tre corpi.

Altro argomento di numerose ed interessantissime pubblicazioni fu la teoria analitica delle superficie, rimessa allora in onore da una celebre Memoria di Gauss, che era passata per lungo tempo inosservata, ma di cui i geometri riconoscevano finalmente la fondamentale importanza. Il Brioschi prese parte grandissima allo svolgimento (divenuto poi sempre più largo e più complesso) di questa teoria, e vi arrecò più d' un contributo essenziale, tra altro col concetto di coordinate curvilinee tangenziali, da lui primamente adombrato in una nota sulla superficie delle onde.

Non volle rimanere estraneo agli studi di pura geometria, il cui decisivo risveglio risale a un dipresso alla medesima epoca, benchè l' indole peculiare del suo ingegno lo chiamasse di preferenza alle ricerche di pura analisi; e fu felicissimo nella trattazione di quelle questioni in cui l' una e l' altra disciplina gareggiano nel raggiungere una stessa meta, del che basterà citare l' esempio fornito dai poligoni di Poncelet.

Ma l' indirizzo in cui il Brioschi si slanciò con vera passione e con istraordinario successo fu quello delle ricerche sulle equazioni algebriche e sui nuovi algoritm che si riassumono nell' uso sistematico dei determinanti, degli invarianti, dei covarianti e delle forme algebriche e simboliche. Basterebbe già il libro dei Determinanti, che risale ai primissimi anni della sua carriera scientifica e che fu tradotto in pressochè tutte le lingue colte, per constatare le eminenti sue doti d' assimilazione e d' invenzione, como il profondo e sicuro possesso d' ogni più disparato dominio dello scibile matematico. Ma sarebbe impossibile analizzare anche sommariamente, senza entrare in particolari troppo disformi dall' indole d' un giornale, l' infinita copia di nuove vedute, di nuove proposizioni, di nuovi procedimenti che si trovano disseminati nelle numerosissime Memorie di lui sulle indicate teorie, fra le quali basterà menzionare quella Monografia sulle forme binarie che doveva riassumere gran parte dei suoi studii e che,

sebbene rimasta incompieta, contiene pur tuttavia un ricco tesoro di materiali preziosi. E, per citare almeno uno dei moltissimi argomenti speciali in cui maggiormente brillarono l' acume e la genialità del Brioschi, giovi ricordare le sue elegantissime ricerche sulle serie analoghe a quella di Sturm.

Per ciò poi che spetta alla dottrina delle equazioni algebriche, basti il dire che, nella memorabile scoperta della risoluzione dell' equazione di 5° grado, il nome di Brioschi è indissolubilmente legato a quelli di Hermite e di Kronecker, con questo di più, ch' egli non ha poi mai cessato di illustrare con nuove ricerche questo campo cosi irto di difficoltà, preparando il terreno e partecipando attivamente ad altri non meno cospicui progressi.

Un altro larghissimo campo di studii ai quali, non meno che ai precedenti, il Brioschi si trovò spontaneamente attratto dalle sue peculiari attitudini e preferenze scientifiche, e che del resto si collegava necessariamente ed intimamente coll' ultimo dei dianzi accennati, fu quello delle funzioni trascendenti, inaugurato da Legendre e recato d' un tratto a smisurate altezze dai lavori di Abel, di Jacobi e da quelli, allora recentissimi, di Weierstrass. Qui forse, più che altrove, il Brioschi era destinato a raccogliere una messe oltremodo feconda e rigogliose, la materia prestandosi mirabilmente a quel suo genio, ormai maturo, di analista supremamente classico, e già egli era entrato gloriosamente nell' arringo, ispirandosi ai lavori di Weierstrass, quando, sopravvenuti gli eventi del 1859, si trovò d' un tratto chiamato a spendere in altro modo le forze esuberanti del suo ingegno. Cosi si chiuse il periodo eroico della sua operosità scientifica, periodo che durò non più d' un decennio, ma che bastò a circondare per sempre il suo nome d' una aureola di gloria purissima così presso di noi, come presso tutte le culte nazioni, di cui in così breve tempo egli aveva saputo assimilare ed eguagliare la poderosa produzione scientifica.

Se tuttavia, in tutto il tempo successivo, egli non potè mai più consacrare alla scienza pura l' intera somma delle sue smisurate energie intellettuali, neppur cessò mai di tener sempre ed amorosamente fiso in essa lo sguardo, tornando ad ogni tratto, e più d' una volta abbastanza intensamente, al culto di essa, così da aggiungere molto al moltissimo già prodotto, e nulla trascurando di ciò che poteva, direttamente ed indirettamente, favorire la diffusione ed il progresso degli alti studii nel nostro paese. In quest' ultimo senso *merita* principalmente d' esser ricordata l' opera indefessa da lui data *nel mantenere* in vita dapprima, e nel recare poscia a rigogliosa

fioritura quella pubblicazione periodica che s' intitola *Annali di matematica pura ed applicata*, e che da non breve tempo rappresenta degnamente l' Italia fra le congeneri e più apprezzate pubblicazioni d' Europa e d' America. Già fin dal 1858, quando questo periodico sorse in Roma, in continuazione d' un altro più modesto che lo precedette, egli aveva contribuito moltissimo a dargli alimento e notorietà; ma nel 1867, quando la vita ne era divenuta alquanto languida e stentata, egli ne trasportò la sede da Roma a Milano, e ne assunse la direzione, dapprima insieme col collega Cremona, poi, dopo la partenza di questo, da solo. Nei trent' anni trascorsi dopo questo rinnovamento dell' antico periodico romano, ne sono apparsi in luce ben 26 volumi in-4°, ai quali collaborarono tutti i migliori matematici italiani e non pochi fra gli stranieri d' ogni nazione, attivando così anche fra noi quel ricambio d' ospitalità scientifica che già s' era iniziato altrove ed al quale il Brioschi aveva già tanto e così ampiamente contribuito coll' esempio e col consiglio.

È inutile dire a lungo dell' opera data, in un senso più universale, a pro della scienza italiana durante la lunga presidenza dell' Accademia dei Lincei. Niuno ignora come primo pensiero del Brioschi sia sempre stato quello di far convergere i maggiori mezzi possibili all' ampliamento delle pubblicazioni accademiche, così da potervi accogliere, come può ora ben dirsi che avvenge, ogni degna manifestazione degli studii nazionali.

Senonchè l' analisi delle varie, per non dire infinite forme sotto cui si estrinsecò l' ardore inestinguibile del Brioschi per tutto ciò che s' attiene agli studii ed agli studiosi del nostro paese, ardore che andò facendosi sempre più largo e comprensivo di ogni sana manifestazione del pensiero scientifico, mentre da un lato condurrebbe troppo lontano e sconfinerebbe dal campo prefisso, riuscirebbe forse dall' altro ad offuscare un cotal poco, specialmente agli occhi di chi non ebbe la ventura di conoscere da vicino l' uomo, l' immagine integra e distinta che di lui dobbiamo formarci, e che è bene rimanga scolpita indelebilmente nella storia degli intelletti d' Italia. La quale immagine è quella d' un forte campione della schietta stirpe latina, la cui mente, sovranamente equilibrata, fu sempre aperta ad ogni più alta aspirazione ideale come ad ogni più intimo bisogno di vertiginosa attività esterna, e che nell' opera sua, di qualunque natura si fosse, recò invariabilmente la benevolenza, la sicurezza, la serenità e, non ultima attrattiva, quell' amabile scioltezza che, nell' esercizio delle discipline e delle cose severe, lascia come spirare un sottile profumo di squisita artisticità.

Signor Brioschi, who was an Honorary Fellow of the Cambridge Philosophical Society, was President of the Reale Accademia dei Lincei, Rome, and Director of the Istituto Tecnico Superiore of Milan. He was elected an Honorary Foreign Member of our Society May 9th, 1878. His only communications to the Society were "Sur une Propriété du Paramètre de la Transformée Canonique des Formes Cubiques Ternaires," Vol. XII., pp. 58–63 (read January 13th, 1881); "Sur la Transformation des Equations Algébriques," Vol. XX., pp. 127–131 (read February 14th, 1889).*

The Council are indebted to Mr. H. M. Taylor for the following notice of Dr. Frost:—

Percival Frost, the second son of Mr. Charles Frost, F.S.A., a solicitor, of Hull, was born in that town on September 1st, 1817. His schooldays were spent at Beverley and Oakham. In October, 1835, he commenced residence at St. John's College, Cambridge. At Oakham he had shown great aptitude for languages, and had competed on equal terms with James Atlay, afterwards Bishop of Hereford, who was ninth in the Classical Tripos of 1840.

At Cambridge, Frost devoted his time chiefly to mathematics, and, for three years, ran an almost neck-and-neck race with B. M. Cowie, the present Dean of Exeter. They were elected Fellows of St. John's College on the same day, March 18th, 1839. In 1841 Frost was ordained, and for some years acted as a curate, but he never held any preferment in the Church. In 1841 he vacated his Fellowship at St. John's on his marriage, and settled down in Cambridge to the work of a private tutor. In this work he was at once popular and successful. Among his pupils, the one who rose to the highest eminence in scientific reputation was the late W. K. Clifford; from him Frost was always ready to admit that he had learnt much. Frost held a Mathematical Lectureship at Jesus College for twelve years, and afterwards at King's College for thirty years. The latter College showed their appreciation of his services as a Lecturer by electing him to a Fellowship in 1882.

Frost examined in the Mathematical Tripos for the first time in January, 1846, and for the last time in the divided Tripos which took place in June, 1882, and January, 1883. Altogether Frost examined

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* *Cf.* also *Proc.*, Vol. XI., p. 157, and Vol. XII., p. 215.

ten times, and his record has only been beaten by Dr. Ferrers, who has examined eleven times.

In 1854 Frost published his first work, a treatise on part of Newton's *Principia*, and, in 1863, in conjunction with the late Joseph Wolstenholme, *A Treatise on Solid Geometry*. Each of these works has passed through several editions, and both are still popular text-books. In 1872 Frost published a third work, *A Treatise on Curve-Tracing*, a work of greater originality, of which Professor Chrystal says, in speaking of the reversion of series and of the expansion in power-series of an algebraic function: "The applications are very numerous, for example, to the finding of curvatures and curves of closest contact, and to curve-tracing generally. A number of beautiful examples will be found in that much-to-be-recommended text-book." He was the author of a considerable number of papers, a list of which will be found in the Royal Society's Catalogue.

On December 9th, 1869, Frost was elected a member of this Society. He never served on the Council or contributed a paper to the Society's *Proceedings;* but he acted as a referee on two or three occasions. In 1882 Frost took the degree of Sc.D. at Cambridge.

It would be an injustice to Frost's memory to omit all reference to his wide interests, his many and varied accomplishments, and to the characteristic simplicity of his generous nature. The playful spirit of the man, who was always ready to see the humorous side of every situation and of every subject, peeps out in the prefaces to his books and occasionally in his mathematical papers.

Frost inherited a splendid constitution. When he completed his eightieth year, there seemed to be a probability of his attaining extreme old age. His mother had reached the age of ninety-two, and her uncle, who held the living of Holy Trinity, Hull, for seventy years, died in his ninety-eighth year. A painful disorder, however, which attacked him, proved fatal after six weeks' illness, and Frost died at Cambridge on Trinity Sunday, June 5th, 1898.*

The following Notice of Dr. Hopkinson has been put at the service of the Council by Mr. R. T. Glazebrook:—

John Hopkinson was the eldest son of Mr. Alderman Hopkinson, of Manchester. He was born in 1849, and educated at Lindon Grove

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* An *in memoriam* sketch by Dr. H. A. Morgan was published in the *Cambridge Review* of June 16th, 1898.

School and Queenswood College until the age of fifteen and a-half years. After two and a-half years spent at Owens College, he entered at Trinity College, Cambridge. He graduated as Senior Wrangler in 1871, and was first Smith's Prizeman of his year, The same year he was elected a Fellow of the College. Early in 1872 he became associated with Messrs. Chance Bros. & Co. as engineer at their lighthouse works near Birmingham. After a residence of about six years in that city, he removed to London, and laid the foundations of his large consulting practice as an electrical engineer. He had taken his D.Sc. degree at London University while still at Cambridge. In 1878 he was elected a Fellow of the Royal Society, and in 1890 he received the Royal Medal for his researches in electricity and magnetism. During his work at Messrs. Chance's, Dr. Hopkinson introduced many improvements into lighthouse machinery, and papers dealing with some of these were communicated to the Royal Society; among them may be named that on the group-flashing apparatus described in the *Proceedings* of the Society for 1874. In the year 1877, the first of his papers on the "Residual Charge of the Leyden Jar" was published, and this was followed by others on the "Measurement of Specific Inductive Capacity." The importance in the theory of electro-magnetism of Maxwell's relation connecting the refractive index with the specific inductive capacity of a substance was early apparent to him, and his observations on glass (*Phil. Trans.*, 1878) were made with a view to verifying this. Experiment showed that, for the materials used—various specimens of flint glass—the specific inductive capacity is not equal to the square of the refractive index. These results were contradicted by the work of Gordon (*Phil. Trans.*, 1879, Pt. 1.), and Hopkinson, in 1880 (*Phil. Trans.*, 1881, Pt. 2), repeated his observations, modifying, however, his method so as to use a rapidly alternating electromotive force—in his earlier experiments the electromotive force had been uniform in direction. Differences in the value of the specific inductive capacity were found in some cases; they were, however, small compared with the differences between his own results and those of Gordon, and Hopkinson's experiments showed conclusively that the large difference between them was due to the want of uniformity of the electric field in Gordon's apparatus, which causes errors greater than any one would suspect without actual trial. About the same time, the result of a series of experiments on the specific inductive capacity of liquids was made. These were extended in 1887 (*Proc. Roy. Soc.*, Vol. XLIII., 1887), and it appeared that, while for hydrocarbons and other liquids

the square of the refractive index is equal to the specific inductive capacity, for animal and vegetable oils it is not so. More recent research has fully confirmed the view that the discrepancy arises from the difference in the length of the waves for which the two quantities, inductive capacity and refractive index respectively, were necessarily measured, and Hopkinson's work has been greatly instrumental in establishing the above-mentioned fundamental relation of Maxwell's theory.

In handing the Royal Medal to Hopkinson in 1890, the President (Lord Kelvin) referred at length to his researches on the effect of temperature on the magnetic properties of iron and other metals. These researches were published in the *Phil. Trans.* for 1889. The experiments show that, for small magnetizing forces, the susceptibility of iron increases with rising temperature until a temperature varying from 690° to 870° C. in different specimens is reached. On approaching this critical temperature from below, the rise is very rapid; when the temperature has been passed, the iron ceases almost completely to be magnetic.

But Hopkinson's most important work is connected with the theory of dynamo-electric machinery. In one of the earliest of his papers on this subject (*Proc. Mech. Eng. Instit.*, 1879, p. 246; 1880, p. 266) he defines as the characteristic of a machine the curve which gives the relation between the electromotive force of the armature and the current round the magnets, and he shows how, from this curve, may be deduced "the result in all employments of the machine, whether as a motor, or to produce a current through a resistance, through an electric arc, or in charging accumulators; also the result of varying the winding of the machine, whether in the armature or the magnets."

In 1886 (*Phil. Trans.*, A, p. 331) appeared his paper on "Dynamo-electric Machinery," written in conjunction with his brother, Dr. E. Hopkinson. The paper, to quote from Prof. A. Gray when reviewing a collected edition of Hopkinson's papers on "Dynamo-electric Machinery" (*Nature*, July, 1893, p. 244), "has become justly famous as that in which his enormously useful idea of the magnetic circuit was first employed in a complete and consistent manner to the discussion of the results of experiments on different types of dynamo." One purpose, as the authors state, is to give "an approximately complete construction of the characteristic curve of a dynamo of given form from the ordinary laws of electro-magnetism and the known properties of iron, and to compare the result of such construction with the actual characteristic of the machine." This is done in a

masterly manner, and the results obtained are tested by experiment.

The latter part of the paper contains a method for measuring the efficiency of a dynamo. Two similar machines have their shafts coupled, and are connected electrically; one acts as a generator, the other as a motor. The motor helps to drive the generator which feeds it. If the two machines are similar, the only additional power necessary to drive the combination is the waste in the two dynamos and in the connexions. This is supplied from outside, and, being small compared with the power supplied from the generator, which can be measured electrically, can be determined by a dynamometer with sufficient accuracy. This idea of the Hopkinsons has borne fruit in various methods for testing dynamos and transformers since devised by others. The paper was completed in 1892 by one presented to the Royal Society by Dr. Hopkinson and E. Wilson. Another paper, published in 1884, dealt with the question of running alternating machines in parallel. In 1882 he invented and described the famous three-wire system of electrical distribution.

Hopkinson was the author of numerous other papers of importance, but sufficient has been written to show the fertility of his powers and the magnitude of his work. Moreover, he was known as a practical engineer no less than as a writer. He was connected with some of the most important of electrical undertakings; it may be sufficient to mention the City and South London Railway and the Manchester Electric Supply Station. At the time of his death he was engaged on the tramway systems of Leeds, Liverpool, and other towns. His work is distinguished by its clearness, its brevity, and its precision. Some of his papers are almost too condensed. At meetings of scientific societies, and in public, he did not usually speak at great length, but every word was to the point; he knew his own mind, and expressed his views concisely and clearly. The style in which he describes his experiments may be taken as a model by others; he states briefly the problem to be solved and then the method he intends to follow for its solution. Sufficient detail of each experiment is given to enable the reader to check and verify his results, and all is clearly arranged; there is no unnecessary verbiage, and no repetition. The loss to physical science by his tragic death is difficult for us who knew him to estimate; there are few whose influence was felt in so many fields of activity, few who are devoting themselves more completely to the advancement of their fellows. He was killed, with a son and two daughters, by a fall from the rocks of the Petite Dent de

Veisivi, at Arolla, on August 27 last; how the accident happened no one can say with certainty. Hopkinson himself, his son, and one of his daughters were mountaineers of exceptional skill and experience in rock climbing, and we may feel sure that no ordinary occurrence was the cause of so terrible a catastrophe. The scene in the beautiful cemetery at Territet, where they were buried the following week, will ever live in the minds of those who were present.

Dr. Hopkinson was elected a member of the Society, January 13th, 1873.

The Secretary is indebted to Mr. Joseph Edwards for the following brief notice of Mr. Walter Wren :—

One of the earliest members of our Society has passed away in the person of Mr. Walter Wren, whose election bears date October 16th, 1865.

Wren was the second son of Mr. Richard Wren, of Buntingford, and was born in 1831. He received his early education at Buntingford Grammar School and Elizabeth College, Guernsey, whence he proceeded to Christ's College, Cambridge, where he had for contemporaries a group of distinguished men, amongst whom were C. S. Calverley, Prof. Seeley, Sir Walter Besant, and Prof. Skeat. Wren's college career was blighted by his being attacked, whilst an undergraduate, by spinal disease, which prevented his taking an Honours degree. He, however, graduated in 1859, and proceeded to the M.A. degree in 1862.

His subsequent career has been a remarkable one, the more so in his success having been achieved in spite of a life of continuous suffering. His fame as a private tutor for candidates for the Government services is well known. For more than thirty years he has been engaged in preparing candidates for the Indian and Home Civil Service Examinations, as well as for Colonial Cadetships, the Army, the Woods and Forests Service, the Indian Police Service, &c. So great has been his success in this task that, at the present day, hundreds of the most distinguished members of the State Services owe their education and position to Mr. Wren's establishment.

Mr. Wren was an excellent judge of capacity. He always insisted upon thoroughness of work. He abhorred the acquirement of superficial knowledge in any subject, and would always urge with all the force of his powerful personality the importance of possessing a thorough mastery of all subjects taken up by his pupils.

As though the management of a great institution like his were not sufficient outlet for Wren's energy, we find him standing at the General Election of 1880, for Wallingford, where he was returned by a majority of forty-one. He was subsequently unseated on petition, but made several subsequent attempts to enter Parliament. He was a member of the first London County Council, and one of the founders of the National Liberal Club.

Wren died on August 5th, 1898, from paralysis, with which he had been seized only a week previously.*

Henry Perigal, who died on June 6th, 1898, in the ninety-eighth year of his age, came of a family noted for their longevity. He traced his descent from Sigurd, a Dane, who, after a successful raid into Normandy, in 908, settled there and assumed the name of Perigal. He was in the tenth generation from Gideon Perigal, and his wife Madeline Duval, of Dieppe, who as Huguenots escaped from prison and settled in London. Mr. Perigal's father, who reached the age of ninety-nine, was one of thirteen children, nine of whom reached a great age. Mr. Perigal, who was born April 1st, 1801, was the eldest of six, the youngest of whom, Mr. Frederick Perigal, is now in his eighty-seventh year. As a result of the great age to which his father lived, he was "Henry Perigal, junior," till he was an old man himself. In early life he was a clerk in the Privy Council Office for a short time, and subsequently he held for many years a position in the office of Messrs. Henry Tudor & Son, Stockbrokers, the senior partner of which firm was a personal friend of his. He was a member of the Astronomical, Microscopical, Meteorological, Photographic, and several other scientific Societies, and a very regular attendant at the meetings of all, usually staying to the end, and often being the last to leave; and it may be said that for thirty or forty years, during eight months of the year, he divided his time between daily office work at 29 Threadneedle Street and evening attendance at the meetings of scientific societies, the lectures at the Royal Institution, &c.

His connexion with the Meteorological Society was remarkable: the Society was founded in 1850; in May, 1853, he became Treasurer,

* [A letter from Sir Walter Besant in the *Times*, over the date August 7th, gives an interesting general sketch of Wren's work, founded upon "an almost life-long friendship."]

and retained this office up his death—for forty-five years—signing cheques and attending to the financial business of the Society almost to the last. When our Society was formed he would have liked to join it at once, but feared that his paradoxical views on the subject of astronomical motions would disqualify him; he was much pleased therefore when, some years later, De Morgan offered to propose him as a member and signed his nomination form.

As a young man Mr. Perigal was much interested in all kinds of lathe work, but especially in curve-tracing by means of Ibbetson's geometric chucks. To this subject he brought to bear great patience, perseverance, and delicacy of manipulation. Many of the curves so drawn by him between 1830 and 1840 are exquisite examples of fine geometrical tracings performed by means of the bow pen. He afterwards delineated mechanically a great number of bicircloidal curves, *i.e.*, curves described by two circular motions. These curves were generated by a point moving uniformly round a circle the centre of which also moved uniformly round another circle. The curve traced depended upon the ratio of the angular velocities in these two circles, this ratio being fixed during the description of the same curve. He also devised little apparatuses, which he called kinescopes, for exhibiting these curves to the eye as they were formed. The curve was traced by a bright point which, by a system of multiplying wheels, was made to move so rapidly as to exhibit a continuous bright line. He also occupied himself with curves generated by more than two circular motions, and as early as about 1835 had delineated the curves now generally known by the name of Lissajous, including the finite or retrogressive parabola, which he had obtained by points some years before he described it mechanically. He also displayed great ingenuity in proving geometrical theorems by means of dissection and transposition of parts. These dissections were mostly discovered between 1830 and 1840. Among the best are his proof of Euclid I. 47, and his decomposition of a square into three squares. Unfortunately nearly all his work was published merely for private distribution among his friends, and, as he had not the pen of a ready writer (as he himself used to say), his diagrams were generally issued without sufficient explanation, and sometimes without any at all. These fugitive pieces, which were being produced by him for sixty years or more, are very numerous. They consist principally of diagrams of bicircloidal curves (some on large sheets), of the "Lissajous" curves, and of dissections. Some of the dissections he made as puzzles, forming

them in card and wood, and gave to his friends. He also engraved them (as well as the finite parabola) upon some of his visiting cards. His proof of Euclid I. 47 was reproduced in Vol. II. of the *Messenger of Mathematics*.

Mr. Perigal's mechanical work in connexion with the lathe led him astray with respect to the motions of the moon and planets. As the Moon always turns the same face to the Earth, he considered that she should be regarded as subject to one simple circular motion, and not (as mathematicians said) to a motion of revolution round the Earth combined with one of rotation round her own axis. The point in dispute was not, as most people thought, merely a question of definitions, for Mr. Perigal assumed as an axiom that a central force necessarily acted upon a body describing a circular orbit round it exactly as if the one was attached to the other by a rigid bar. The "Moon paradox" was afterwards taken up by Mr. Jellinger Symons, and is frequently connected with his name. Mr. Perigal made a great many mechanical devices to illustrate his views, and wrote a number of little tracts and fly-sheets, in prose and verse, on "Revolution and Rotation," the "Fallacy of the Moon's Rotation," &c. These were generally circulated privately, but some appeared in the *Astronomical Register*. He was almost unique as a paradoxer in being a *persona grata* wherever he went, and neither at our Society, nor at the Astronomical, did he ever bring forward his own views aggressively or in a manner that was objectionable to those who differed from him. He is included in the "Budget of Paradoxes"; but De Morgan lets him off very gently, saying that "he has published valuable diagrams in profusion," and that he was "entirely indebted to him for the illustrations of the article on 'Trochoidal Curves,' published in the *Penny Cyclopædia*, without which the article would not have been written." There is no reason to doubt that he always believed that the "Perigalian system of Astronomy" would eventually supersede the Newtonian, but this did not interfere with his personal friendships with those who believed in a system which seemed to him obviously absurd. Mr. Proctor's book on the *Geometry of Cycloids* was chiefly illustrated by blocks cut in the lathe by Mr. Perigal; and Northcote in his work on *The Lathe* gives an illustration of the actual apparatus used, together with Perigal's rules for the use of Ibbetson's chucks. It may be mentioned that about 1868 Mr. Perigal publicly offered a prize of £10 for the discovery of a curve (one of his own curves produced by circular motions) satisfying certain conditions. The prize was won by Clerk

Maxwell, who obtained Mr. Perigal's curve, and also by Mr. W. D. Bushell. Many years ago he wrote a paper (in the *Journal of the Society of Arts*) on the way in which great monoliths, like those at Stonehenge, could be moved by such appliances as savages could use, and he also wrote upon the probable methods used in the construction of the Great Pyramid.

He was a warm friend, and a pleasant companion, gentle in manner, amiable, kind-hearted, and generous, interested in the topics of the day, and shrewd without being cynical. He rarely left London, and probably never went abroad in the last half of his life. At some time he had made a voyage to Madeira. Twenty years or more ago he let his beard grow, and since then his refined face, slight and fragile form, and long white beard and hair, made him a striking and picturesque figure. He ceased to attend scientific meetings two or three years ago, and for the last year or more had ceased to leave his lodgings; but there was no failure of mind or eye-sight. He was never married. By his own wish he was cremated at Woking.*

Mr. Perigal was elected a member of our Society, January 23rd, 1868, and was a frequent attendant at our meetings. There is a characteristic note by him, entitled "Link Trammels," in Vol. v., p. 144, in which he claims that Jopling had anticipated Peaucellier. The only other contribution he made was a verbal account of some "Geometrical Metamorphoses by Partition and Transformation" (January 8th, 1891). Many of these figures are to be seen in his pamphlet "Graphic Demonstrations of Geometric Problems," selected by H. Perigal, which he presented to the members of the Association for the Improvement of Geometrical Teaching (1891), of which Association he was elected a member in 1874.

Prof. Burnside sends the following note:—

While the paper "On the Continuous Group that is defined by any Given Group of Finite Order," pp. 546–565, was passing through the press, the author had not the leisure to study carefully the memoir by Herr Frobenius "Über die Darstellung der endlichen Gruppen durch lineare Substitionen," referred to on p. 548. Having done so since, he wishes to acknowledge that the result contained in the concluding paragraph of his paper (p. 565) is anticipated in Herr Frobenius' memoir.

* Dr. Glaisher, in a note which accompanied the above account of his old friend, writes "it was in 1855 that I first saw Perigal."

ERRATA AND ADDENDA.

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Page 103, in result (111), *for* "$\left(\frac{X^2+XY+Y^2}{2}\right)^2$," *read* "$\left(\frac{X^2-2XY+Y^2}{2}\right)^2$."

,, 636, add to Lt.-Col. A. Cunningham's papers: "Note of Two High Primes, p. 328."

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Page 379, line 9 from foot, *for* "of $(3^{105}+1) \div (3^{35}+1)$," *read* "of the algebraic prime factor of $(3^{105}+1)$."

VOL. XXIX.

Page 405, result (53d'), *for* "$4y^4)+5$," *read* "$4y^4)^2+5$."

,, 410, table of factors of $(1+Y^{18})$, line of $Y=18$, col. of L, *insert* "33,388,093; B."

,, 428, col. of M, line of $n=39$, *for* "157 . 1041;" *read* "157 . 10141;"

,, 429, table of factors of $(1+Y^{15})$, col. of M, lirϵ of $Y=675$, *after* "31," *insert* "51,241 . 139,801; C."

,, 432, result (106), *for* "(A^2+3B^2)," *read* "$(A^2+3I^2)^3$."

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